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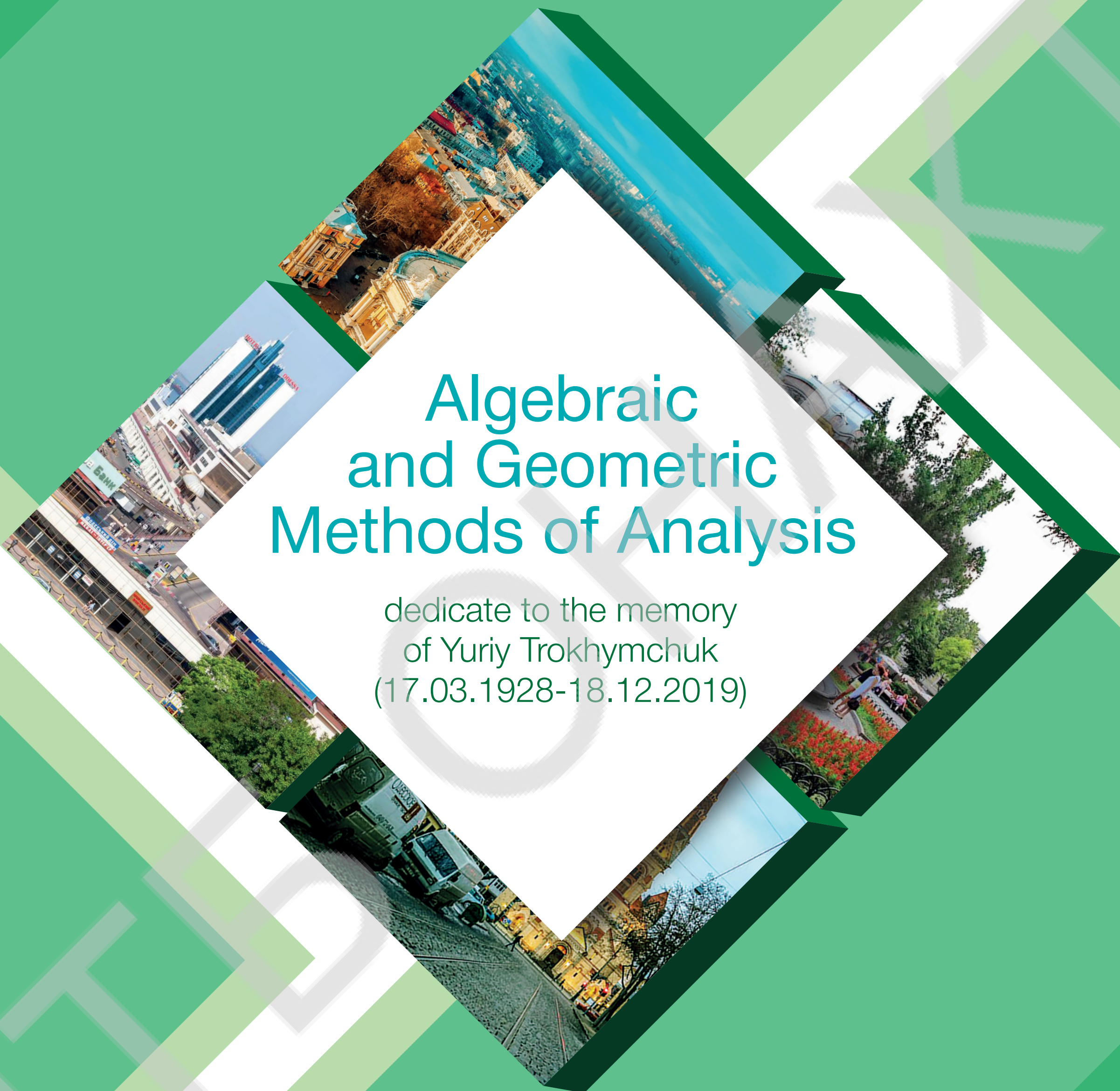

Algebraic and Geometric Methods of Analysis

dedicate to the memory
of Yuriy Trokhymchuk
(17.03.1928-18.12.2019)

May 25-28, 2021
Odesa, Ukraine

List of topics

- Topological methods in analysis
- Geometric problems of complex and mathematical analysis
- Algebraic methods in geometry
- Differential geometry in the whole
- Geometry and topology of differentiable manifolds
- General and algebraic topology
- Geometric and topological methods in natural sciences

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Finitely bi-Lipschitz homeomorphisms between Finsler manifolds

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In this talk we investigate the boundary behavior of finitely bi-Lipschitz homeomorphisms between Finsler manifolds. Our study involves the module technique and classes of mappings whose moduli of the curve/surface families are integrally controlled from above and below. The Lusin (N)-property with respect to the k-dimensional Hausdorff measure for the finitely bi-Lipschitz mappings is also established. The talk is based on a joint work with A. Golberg; see [1].

Let $\mathbb{M}$ be an n-dimensional differentiable manifold, $n \geq 2$. By the differentiability we mean C^∞–differentiability. For a point $x \in \mathbb{M}$, $T_x\mathbb{M}$ denotes the tangent space at x, and $T\mathbb{M} := \cup_{x\in\mathbb{M}} T_x\mathbb{M}$ is the tangent bundle. The *Finsler manifold* is a differentiable manifold $\mathbb{M}$ equipped the Finsler metric $\Phi(x,\xi) : T\mathbb{M} \to \mathbb{R}^+$ satisfying the conditions:

(i) *regularity*: $\Phi \in C^\infty$ on $T\mathbb{M}_0 := T\mathbb{M} \setminus \{0\}$;

(ii) *positive homogeneity*: Φ is positive homogeneous that is $\Phi(x, a\xi) = a\Phi(x,\xi)$ for all positive $a \in R$ and $\Phi(x,\xi) > 0$ for $\xi \neq 0$;

(iii) *the Legendre condition or strong convexity condition*: $g_{ij}(x,\xi) = \frac{1}{2}\frac{\partial^2\Phi^2(x,\xi)}{\partial\xi^i\partial\xi^j}$ is positive definite whenever $\xi \neq 0$.

Following [3], an element of *volume* on the Finsler manifold is defined by $d\sigma_\Phi(x) := \frac{|B^n|}{|B^n_x|}\, dx^1...dx^n$, where $|B^n|$ denotes the Euclidean volume of the unit n-ball whereas $|B^n_x|$ is the Euclidean volume of the set $B^n_x = \left\{(\xi^1,...,\xi^n) \in \mathbb{R}^n : \Phi\left(x, \sum\limits_1^n(\xi^i, e_i(x))\right) < 1\right\}$ with an arbitrary basis $\{e_i(x)\}_{i=1}^n$ in $\mathbb{R}^n$ depending on x.

Suppose that D and D' are two domains on $\mathbb{M}$ and $\mathbb{M}'$, respectively, $f : D \to D'$ is a continuous mapping. Let $L(x,f) = \limsup_{y\to x}\frac{d_{\Phi'}(f(x),f(y))}{d_\Phi(x,y)}$, $x \in D$ and $l(x,f) = \liminf_{y\to x}\frac{d_{\Phi'}(f(x),f(y))}{d_\Phi(x,y)}$.

Following [2], we say that $f : D \to D'$ is *finitely Lipschitz* if $L(x,f) < \infty$ for all $x \in D$ and *finitely bi-Lipschitz* if

$$0 < l(x,f) \leq L(x,f) < \infty$$

for all $x \in D$.

A Borel function $\rho : \mathbb{M} \to [0,\infty]$ is called *admissible* for the family Γ of k-dimensional surfaces S in $\mathbb{M}$, $k = 1,\ldots,n-1$, (abbr. $\rho \in \operatorname{adm}\Gamma$), if

$$\int\limits_S \rho^k\, d\mathcal{A}_\Phi \geq 1, \quad \forall\, S \in \Gamma. \tag{1}$$

Following [2], the function $\rho : \mathbb{M} \to [0,\infty]$ measurable with respect to the measure of a volume σ_Φ is called *extensively admissible* for a family Γ of k-dimensional surfaces S in $\mathbb{M}$ (abbr. $\rho \in \operatorname{ext\,adm}\Gamma$), if the admissibility condition (1) holds for almost all (a.a.) $S \in \Gamma$.

The *conformal module* or *module* (called also the conformal modulus) of a family Γ of k-dimensional surfaces in D is defined by

$$M(\Gamma) := \inf_{\rho\in\operatorname{adm}\Gamma}\int\limits_D \rho^n(x)\, d\sigma_\Phi(x),$$

where D is a domain in $\mathbb{M}$.

Let $Q:\mathbb{M}\to(0,\infty)$ be a measurable function. A homeomorphism $f:D\to D'$ is called *lower Q-homeomorphism* at a point $x_0\in\overline{D}$, if there exists $\delta_0\in(0,d(x_0))$, $d(x_0):=\sup\limits_{x\in D} d_\Phi(x,x_0)$, such that for any $\varepsilon_0<\delta_0$ and any geodesic rings $A_\varepsilon=A(x_0,\varepsilon,\varepsilon_0)=\{x\in\mathbb{M}:\varepsilon<d_\Phi(x,x_0)<\varepsilon_0\}$, $\varepsilon\in(0,\varepsilon_0)$, the inequality

$$M(f(\Sigma_\varepsilon)) \;\geq\; \inf_{\rho\in\mathrm{ext\,adm}\,\Sigma_\varepsilon} \int\limits_{D\cap A_\varepsilon} \frac{\rho^n(x)}{Q(x)}\, d\sigma_\Phi(x) \tag{2}$$

holds. Here Σ_ε stands for the family of all intersections of the geodesic spheres $S(x_0,r)=\{x\in\mathbb{M}: d_\Phi(x,x_0)=r\}, r\in(\varepsilon,\varepsilon_0)$, with the domain D. We say that the homeomorphism $f:D\to D'$ is a *lower Q-homeomorphism* in D, if f is lower Q-homeomorphism at every point $x_0\in\overline{D}$.

For sets A,B and C, we denote by $\Delta(A,B;C)$ the set of all curves $\gamma:[a,b]\to\mathbb{M}$, which join A and B in C, i.e. $\gamma(a)\in A$, $\gamma(b)\in B$ and $\gamma(t)\in C$ for all $t\in(a,b)$.

Let $Q:\mathbb{M}\to(0,\infty)$ be a measurable function. We say that a homeomorphism $f:D\to D'$ is *ring Q-homeomorphism at a point $x_0\in\overline{D}$*, if

$$M\left(\Delta(f(K),f(K_0);D')\right) \;\leq\; \int\limits_{D\cap A_\varepsilon} Q(x)\cdot\eta^n\left(d_\Phi(x,x_0)\right) d\sigma_\Phi(x) \tag{3}$$

holds for any geodesic ring $A_\varepsilon=A(x_0,\varepsilon,\varepsilon_0)$, $0<\varepsilon<\varepsilon_0<\infty$, any two continua (compact connected sets) $K\subset\overline{B(x_0,\varepsilon)}\cap D$ and $K_0\subset D\setminus B(x_0,\varepsilon_0)$ and each Borel function $\eta:(\varepsilon,\varepsilon_0)\to[0,\infty]$, such that $\int\limits_{\varepsilon}^{\varepsilon_0}\eta(r)dr=1$. We say that f is *a ring Q-homeomorphism in D*, if (3) holds for all points $x_0\in\overline{D}$.

Recall that a metric space $\mathbb{M}$ is called *hyperconvex* if $\cap_{\alpha\in\Lambda}\overline{B}(x_\alpha,r_\alpha)\neq\emptyset$ for any collection of points $\{x_\alpha\}_{\alpha\in\Lambda}$ in $\mathbb{M}$ and positive numbers $\{r_\alpha\}_{\alpha\in\Lambda}$ such that $d(x_\alpha,x_\beta)\leq r_\alpha+r_\beta$ for any α and β in Λ.

The main result of talk is following

Theorem 1. ([1]) *Let D and D' be two domains in Finsler n-dimensional manifolds $(\mathbb{M},\Phi)$ and $(\mathbb{M}',\Phi')$, respectively, $n\geq 2$, and let $\mathbb{M}'$ be a hyperconvex space. If $f:D\to D'$ is a finitely bi-Lipschitz homeomorphism then f is both lower Q-homeomorphism with $Q=K_I^{\frac{1}{n-1}}(x,f)$ and ring Q_*-homeomorphism with $Q_*=C\cdot K_I(x,f)$, where $K_I(x,f)\in L^1_{loc}$ stands for the inner dilatation of mapping f, and C is a constant arbitrarily close to 1.*

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About longest and shortest chords passing through a fixed point

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A new method to construct a tangent to the conchoid of Nicomedes or limaçon of Pascal curves is discussed. Some interesting properties of the cardioid curve (which is a special case of limaçon of Pascal) are investigated. The following problem is studied: "Given a line k and two points A and B on one side of k, find point C such that the sum of lengths of segments CD and CE is minimal, where D and E are intersections of line k with lines CA and CB, respectively". This problem is dual to the classic problem to find shortest segment inscribed to a given angle and passing through a given point. Part of this problem was solved and the remaining part is left as an open question. The problem to find ellipse's longest or shortest chord passing through a given point, is also considered. For the solution the curve named as ophiuride is used.

The following Lemma is used.

Lemma 1. *Let c_1 and c_2 be two arbitrary smooth curves. Let O be a given point and let a line through this point intersect the curves c_1 and c_2 at points A and B. If the length of segment AB is maximal/minimal or constant and the tangents to the curves c_1 and c_2 at points A and B are not perpendicular or parallel to the line AB then these tangents intersect at a point C such that for the perpendicular CD of the line AB the equality $|OA| = |BD|$ holds true.*

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Some equivariant properties of Milnor's construction

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In 1953 John Milnor, for a topological group G, introduced the notion of an infinite join $E_G = G * G * \ldots$. This space possesses a natural action of the group G under which it becomes a universal principal G-fibration. The orbit space $B_G = E_G/G$ is well known as a classifying space. In this talk I will present a more transparent approach to constructing of E_G that will allow us to show that the natural action $G \curvearrowright E_G$ is proper in the sense of R. Palais whenever G is a locally compact group. As a result we obtain some new equivariant properties of this classic space. Similar research is carried out for the complete infinite join $\widetilde{E}_G$ (which is the completion of E_G with respect to a suitable metric) introduced in 1992 by T. Banakh.

Commuting sets for topological set operators

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Let X be a set and $F, G : 2^X \to 2^X$ be two set operators on X. We say that a set $A \subset X$ is *commuting set* for the pair F, G if $F(G(A)) = G(F(A))$.

For a topological space X commuting sets for the pair of set operators Cl, Int were characterized by Levine [2] as symmetric differences of clopen sets with nowhere dense sets. Similarly, Staley [3] obtained a criterion for commuting sets for the pair Int, ∂ (here ∂ denotes the topological boundary operator).

In this work we consider the following six set operators on a topological space: Cl, Int, ∂, Ext (the exterior of a set), $*$ and $+$: $A^* = A \backslash IntA$, $A^+ = ClA \backslash A$ (these two operators were explicitly defined and studied by Elez and Papaz [1]). It is possible to obtain characterizations of commuting sets for each pair of these six operators. As an application of these characterizations we present new criteria for the following well-known classes of topological spaces:

- *nodec*: a space in which every nowhere dense set is closed;
- *extremally disconnected*: a space in which the closure of every open set is also open;
- *strongly irresolvable*: a space in which each open subspace is *irresolvable* (i.e. it cannot be expressed as a disjoint union of two dense sets);
- *perfectly disconnected*: a T_0-space in which any pair of disjoint subsets have no common limit points.

Theorem 1. *Let B be a clopen set and C be a nowhere dense set. Then the symmetric difference $B \triangle C$ is a commuting set for the pair $Cl, *$ if and only if $B \cap C$ is closed.*

Corollary 2. *A space is nodec if and only if any commuting set for the pair Cl, Int is also a commuting set for the pair $Cl, *$.*

Proposition 3. *Let X be a space. Then:*

(1) *X is extremally disconnected if and only if any open set is a commuting set for the pair Cl, Int;*
(2) *X is strongly irresolvable if and only if any nowhere dense set is a commuting set for the pair Cl, Int.*

Corollary 4. *A space is extremally disconnected and strongly irresolvable if and only if any set is a commuting set for the pair Cl, Int.*

Proposition 5. *A space is perfectly disconnected if and only if any set is a commuting set for the pair $Cl, *$.*

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Asymptotic analysis of quasi-regular mappings in space

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Dedicated to the memory of Yuri Yu. Trokhymchuk

We investigate quasisymmetric embeddings of a closed 3-ball inextensible in neighborhoods of any boundary points and bounded quasiregular locally homeomorphic mappings in 3-space, especially their behavior in the unit 3-ball and their asymptotics while approaching the boundary of the unit 3-ball(cf. [12], [14], [7], [16], [17]).

We discover several new properties of such mappings in 3-space. Such discoveries are based on our construction of a new type of bounded locally homeomorphic quasiregular mappings in 3-sphere (and in the unit 3-ball)- see [6]. It addresses long standing problems for such mappings, including M.A.Lavrentiev problem, Pierre Fatou problem and Matti Vuorinen injectivity and asymptotics problems (cf. [7]). The construction of such mappings comes from our construction of non-trivial compact 4-dimensional cobordisms M with symmetric boundary components and whose interiors have complete 4-dimensional real hyperbolic structures (cf. [4]). Such bounded locally homeomorphic quasiregular mappings are defined in the unit 3-ball $B^3 \subset \mathbb{R}^3$ as mappings equivariant with the standard conformal action of uniform hyperbolic lattices $\Gamma \subset \operatorname{Isom} H^3$ in the unit 3-ball and with its discrete representation $G = \rho(\Gamma) \subset \operatorname{Isom} H^4$ (cf. [6]). Here G is the fundamental group of our non-trivial hyperbolic 4-cobordism $M = (H^4 \cup \Omega(G))/G$ and the kernel of the discrete representation $\rho\colon\Gamma \to G$ acould be a free group F_m on arbitrary large number m generators.

Such discrete non-faithful representations of hyperbolic lattices with arbitrarily large kernel were known only for non-uniform case due to the W.Thurston's non-rigidity theorem (Dehn surgeries on cusp ends of non-compact hyperbolic 3-manifolds). We are able to present our construction for uniform (co-compact) hyperbolic 3-lattices based on a new effect in the theory of deformations of hyperbolic 3-manifolds/orbifolds or their uniform hyperbolic lattices $\Gamma \subset \operatorname{Isom} H^3$ (i.e. in the Teichmüller spaces of conformally flat structures on closed hyperbolic 3-manifolds -cf. [1, 2]). We show that such Teichmüller space or the corresponding variety of conjugacy classes of discrete representations $\rho\colon\Gamma \to \operatorname{Isom} H^4$ may have connected components whose dimensions differ by arbitrary large numbers -cf. [3, 5]. This is based on our enhancement to the conformal category of the Gromov-Piatetski-Shapiro interbreeding construction [13] and our construction of non-trivial "symmetric hyperbolic 4-cobordisms" [8].

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Generalized (σ, τ)-derivations on associative rings satisfying certain identities

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The commutativity of associative rings with derivations have become one of the focus points of several authors and a significant work has been done in this direction during the last two decades. It represents the answer to the natural questions of Ring Theory which reach to determine the conditions implying commutativity of the ring. Basically, the study of derivation was initiated during the 1950s and 1960s. Derivations of rings got a tremendous development in 1957, when Posner [1] established two very striking results in the case of prime rings. A considerable amount of work has been done on derivations and related maps during the last decades (see, e.g., [2,3 and 4] and references therein).The main purpose of this paper is present results concerning generalized (σ, τ)-derivations via associative rings. Accurately, we prove the commutativity with other cases of a ring that satisfied certain conditions. These results are in the sprite of the well-known theorem of the commutativity of prime and semiprime rings with generalized derivation satisfying certain polynomial constraints. Throughout this paper, R always represents an associative ring and $Z(R)$ is its center. Let σ and τ be two mappings from R to itself. For any $x, y \in R$ we write $[x, y]_{(\sigma,\tau)}$ for the commutator $x\sigma(y) - \tau(y)x$ and $(x \circ y)_{(\sigma,\tau)}$ for anti-commutator $x\sigma(y) + \tau(y)x$.
Recall that R is semiprime if $aRa = 0$ implies $a = 0$ and R is prime if $aRb = 0$ implies $a = 0$ or $b = 0$. Every prime ring is semiprime ring but the converse is not true always. An additive mapping $d : R \longrightarrow R$ is said to be an (σ, τ)-derivation of R if $d(xy) = d(x)\sigma(y) + \tau(x)d(y)$ holds for $x, y \in R$. Let σ and τ be endomorphisms of R. An additive mapping $D : R \longrightarrow R$ is said to be a generalized (σ, τ)-derivation of R if there exists an (σ, τ)-derivation $d : R \longrightarrow R$ of R such that $D(xy) = D(x)\sigma(y) + \tau(x)d(y)$ for all $x, y \in R$.

Theorem 1. *Let R be a non-zero semiprime ring with nonzero commutator, σ and τ be automorphsim mappings. If R admits a generalized (σ, τ)-derivation satifises the identity $D(x)oy = D(xy)$ for all $x, y \in R$, then $D = 0$.*

Theorem 2. *Let R be a 2-torsion free semiprime ring with nonzero commutator, σ and τ be automorphsim mappings. If R admits a generalized (σ, τ)-derivation satisfies the identity $D(xoy) = D(x)oy - D(y)ox$ for all $x, y \in R$, then $d = 0$.*

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The Tucker HO-SVD and the anisotropy of Finslerian geometric models

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The tensor spectral theory, as an extension of the classic spectral theory of linear operators reached sound applications in Big Data and Image Processing, based on the main decomposition tools of Tucker type, originating in the cornerstone HO-SVD decomposition [7, 8]. This theory enhances the statistical analysis in various fields, originally including MRI-imaging, Special Relativity, ecology, and HARDI biology. The present talk provides a brief survey of results recent tensor spectral theory, and of its applications to geometric structures which rely on *anisotropic metrics of Finsler type*. Several models are addressed, for which we derive the Tucker type HO-SVD decomposition and the induced powerful approximation provides aids for identifying main geometric features, and consistent anisotropy estimates for the Finslerian structures.

We include the mth candidate models for Special Relativity (Pavlov-Chernov, Bogoslovsky, and Roxbourgh models), for ecology (P.L. Antonelli &al.), HARDI biology (L. Astola & al.), Garner oncology and the physics of Langmuir-Blodgett monolayers.

We also present a brief survey of results from the spectral theory of covariant main symmetric tensors which rely on the fundamental tensor fields of the anisotropic geometric models. We note that the spectral data describe properties of the indicatrices associated to the Finsler norms, point out their asymptotic properties, and allow to derive best rank-I approximations - which provide simpler consistent estimates for the original anisotropic structures. We investigate the spectral data of covariant symmetric tensor fields, and focus on the metric and Cartan fields of the Finsler structures - including the Euclidean and Riemannian subcases - and further provide and discuss natural alternatives of the spectral equations.

We consider $n-$dimensional Finsler structures (M, F) with the main axioms relaxed by either dropping the positivity condition, or reducing the domain, and replacing the positive-definiteness of the Finsler metric d-tensor field with the non-degeneracy and constant signature condition. We shall denote by $g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ and $C_{ijk} = \frac{1}{4}\frac{\partial^3 F^2}{\partial y^i \partial y^j \partial y^k}$ the components of the metric and Cartan d-tensor fields, respectively. One of the important features of the Cartan tensor is that its vanishing makes g quadratic in y, and consequently the Finsler space becomes Riemannian (correspondingly, pseudo-Finsler spaces become, in such case, pseudo-Riemannian).

For a real m-covariant symmetric tensor field T on the flat manifold $V = \mathbb{R}^n$ endowed with the Euclidean metric g, we say that a real λ is a Z-eigenvalue and that a vector y is an Z-eigenvector associated to λ, if they satisfy the system:

$$T.y^{m-1} = \lambda \cdot y, \quad g(y, y) = 1, \qquad \text{where} \quad T.y^{m-1} = \sum_{i,i_2,\ldots,i_m \in \overline{1,n}} T_{i,i_2 \ldots i_m} y_{i_2} \cdot \ldots \cdot y_{i_m} dx^i,$$

where by lower dot is repeated transvection and the power is tensorial. As well, an alternative for spectral objects is the H-eigenvalue λ and its H-eigenvector, described by the homogeneous polynomial system: $(T.y^{m-1})_k = \lambda (y^k)^{m-1}$. Regarding the spectra consistency, it is known that in the Euclidean subcase, the Z- and the H-spectra are nonempty for even symmetric tensors, and that a symmetric tensor T is positive defynite/semi-definite iff all its $H-$ (or $Z-$) eigenvalues are positive/non-negative.

Among the applications of the spectral Finsler approach, there was observed the relevance of eigendata of the metric and Cartan tensors of the Langmuir-Finsler structure [1, 5, 9] within the physics of monolayers which studies the interphase boundary of a mono-molecular system.

For the Langmuir model, the Cartan tensor

$$C = C_{ijk}(x, y_*)dx^i \otimes dx^j \otimes dx^k, \quad C_{ijk}(x, y) = \frac{1}{4}\frac{\partial^3 F^2}{\partial y^i \partial y^j \partial y^k}, \tag{1}$$

considered at the fixed supporting element $(x, y_*) \in \widetilde{TM}$, has the 1-mode provided by the slices

$$C = (C_{1ij} = \gamma \cdot M, \;\; C_{2ij} = \nu \cdot M, \;\; C_{3ij} = O_{3\times 3}),$$

where $\gamma = \frac{3A}{\beta^3}$, $\nu = -\frac{\alpha}{\beta}\gamma$, $y_* = (\alpha, \beta, \gamma) \in T_x M$ is the supporting element, and $M = \begin{pmatrix} \beta^2 & -\alpha\beta & 0 \\ -\alpha\beta & \alpha^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Theorem 1. *Consider the Cartan tensor of the Cartan-Langmuir tensor* (1) *from above. Then the associated spectral data are given as follows*

a) The Z-eigendata are given by:

$$S_{\lambda_1=0} = \left\{ \frac{1}{\sqrt{\alpha^2+\beta^2+t^2}}(\alpha,\beta,t) \,\middle|\, t \in \mathbb{R} \right\}, \quad S_{\lambda_2 = \frac{9A^2}{\beta^8\sqrt{\alpha^4+\beta^4}}} = \left\{ \frac{\pm 1}{\sqrt{\alpha^4+\beta^4}}(\beta^2, \alpha^2, 0) \right\} \ni v_\pm .$$

b) The H-eigendata are the following:

$$S_{\lambda_1=0} = \left\{ \frac{1}{\sqrt{\alpha^2+\beta^2+t^2}}(\alpha,\beta,t) \,\middle|\, t \in \mathbb{R} \right\}, \quad S_{\lambda_2 = \frac{9A^2(\beta^2-\alpha^2)^2}{\beta^8}} = \left\{ \pm \left(\frac{\beta}{\sqrt{\alpha^2+\beta^2}}, \frac{-\alpha}{\sqrt{\alpha^2+\beta^2}}, 0 \right) \right\}.$$

Corollary 2. *The Candecomp approximation of the Langmuir-Cartan tensor* (1) *is twofold:*

$$C \sim A = \lambda_2 \cdot v_\pm \otimes v_\pm \otimes v_\pm, \quad v_\pm \in S_{\lambda_2}.$$

Moreover, the HOSVD decomposition and the partial/total ranks of the Cartan tensor are shown to be relevant in estimating the anisotropy level of the direction-dependent Finsler structure. We also note that while the Z-eigenproblem allows a globally covariant alternative, the H-eigenproblem exhibits a strongly local character.

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Invariant structures on homogeneous Φ-spaces and Lie groups

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Homogeneous Φ-spaces were first introduced by V.I. Vedernikov in 1964. Fundamental results for regular Φ-spaces and, in particular, homogeneous k-symmetric spaces were obtained by N.A. Stepanov, A. Ledger, A. Gray, J.A. Wolf, A.S. Fedenko, O. Kowalski and others. It turned out that homogeneous k-symmetric spaces G/H admit a commutative algebra $\mathcal{A}(\theta)$ of *canonical* structures [2]. The remarkable feature of these structures is that all of them are invariant with respect to both the Lie group G and the generalized "symmetries" of G/H. The classical example is the canonical almost complex structure J on homogeneous 3-symmetric spaces with its many applications (N.A. Stepanov, A. Gray, V.F. Kirichenko, S. Salamon and others). For $k > 3$ the algebra $\mathcal{A}(\theta)$ contains a large family of classical structures such as almost complex ($J^2 = -id$), almost product ($P^2 = id$), f-structures of K. Yano ($f^3 + f = 0$) and some others [2]. We dwell on several applications of canonical structures as well as on left-invariant structures on nilpotent and solvable Lie groups.

1) *The generalized Hermitian geometry* (V.F. Kirichenko, D. Blair, S. Salamon and others): canonical nearly Kähler, Killing, Hermitian metric f-structures on homogeneous k-symmetric spaces [2], [3]; left-invariant nearly Kähler and Hermitian f-structures on some classes of nilpotent Lie groups (especially, on 2-step nilpotent and some filiform Lie groups [4]); on generalized (in various senses) Heisenberg groups in dimension 5, 6 [5], and 8; on special solvable Lie groups (group of hyperbolic motions of the plane and its generalizations, the oscillator group and some others); heterotic strings.

2) *Homogeneous Riemannian geometry*: the Naveira classification of Riemannian almost product structures; canonical distributions on Riemannian homogeneous k-symmetric spaces; the classes **F** (foliations), **AF** (anti-foliations), **TGF** (totally geodesic foliations); the Reinhart foliations [2].

3) *Elliptic integrable systems*: homogeneous k-symmetric spaces and associated elliptic integrable systems; a new generalization of almost Hermitian geometry; a new contribution to nonlinear sigma models (F. Burstall, I. Khemar [7]).

4) *Metallic structures*: so-called metallic structures (golden, silver and others), which are fairly popular (especially, golden structures) in many recent publications (M. Crasmareanu, C.-E. Hretcanu [8], A. Salimov, F. Etayo and others); canonical structures of golden type on homogeneous k-symmetric spaces [9].

5) *Symplectic geometry*: bi-Poisson geometry and bi-Hamiltonian systems [10], Hamiltonian vector fields and integrable almost-symplectic Hamiltonian systems [11], canonical almost symplectic structures on Riemannian homogeneous k-symmetric spaces.

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Every 2-dimensional Banach space has the Mazur-Ulam property

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A Banach space X is defined to have the Mazur–Ulam property if for every Banach space Y every isometry $f : S_X \to S_Y$ between the unit spheres of X, Y extends to a linear isometry of the spaces X, Y. In 1987 Tingley posed a (still open) problem if every Banach space has the Mazur-Ulam property. It has been shown that many classical Banach spaces (like $C(K)$, $\ell_p(\Gamma)$, $L_p(\mu)$) do have the Mazur-Ulam property. The main result of the talk is the following solution of the Tingley problem in dimension 2.

Theorem 1. *Every 2-dimensional Banach space has the Mazur–Ulam property.*

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A connection between L-index of vector-valued entire function and L-index of each its component

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The present talk is devoted to the properties of entire vector-valued functions of bounded L-index in join variables. We need some notations and definitions. Let $\mathbf{L} : \mathbb{C}^n \to \mathbb{R}^n_+$ be any fixed continuous function. We consider a class of vector-valued entire functions $F = (f_1, \dots, f_p) \colon \mathbb{C}^n \to \mathbb{C}^p$. For this class of functions there was introduced a concept of boundedness of $\mathbf{L}$-index in joint variables.

Let $\|\cdot\|_0$ be a norm in $\mathbb{C}^p$. Let $\mathbf{L}(z) = (l_1(z), \dots, l_n(z))$, where $l_j(z) \colon \mathbb{C}^n \to \mathbb{R}_+$ is a positive continuous function. An entire vector-valued function $F \colon \mathbb{C}^n \to \mathbb{C}^p$ is said to be of bounded $\mathbf{L}$-index in joint variables, if there exists $n_0 \in \mathbb{Z}_+$ such that $(\forall z = (z_1, \dots, z_n) \in \mathbb{C}^n)(\forall J \in \mathbb{Z}^n_+)$:

$$\frac{\|F^{(J)}(z)\|_0}{J!\mathbf{L}^J(z)} \le \max\left\{\frac{\|F^{(K)}(z)\|_0}{K!\mathbf{L}^K(z)} \colon K \in \mathbb{Z}^n_+, \|K\| \le n_0\right\},$$

where $F^{(J)}(z) = (f_1^{(J)}(z), \dots, f_p^{(J)}(z)), f_k^{(J)}(z) = \dfrac{\partial^{\|J\|}}{\partial z_1^{j_1} \dots \partial z_n^{j_n}} f_k(z)$, $\|J\| = j_1 + \dots + j_n$, $J! = j_1! \cdot \ldots \cdot j_n!$ for $J = (j_1, \dots, j_n)$, $k \in \{1, \dots, p\}$. The least such integer n_0 is called the $\mathbf{L}$-index in joint variables and is denoted by $N(F, \mathbf{L})$.

Denote by $\mathbb{D}^n[z_0, R/\mathbf{L}(z_0)] = \big\{z = (z_1, \dots, z_n) \in \mathbb{C}^n \colon |z_j - z_{j0}| \le r_j/l_j(z_0)$ for every $j \in \{1, \dots, n\}\big\}$ the closed polydisc in $\mathbb{C}^n$. Let Q^n be a class of continuous functions $\mathbf{L} \colon \mathbb{C}^n \to \mathbb{R}^n_+$ such that $0 < \lambda_{1,j}(R) \le \lambda_{2,j}(R) < \infty$ for any $j \in \{1, 2, \dots, n\}$ and $\forall R = (r_1, \dots, r_n) \in \mathbb{R}^n_+$, where $\lambda_{1,j}(R) = \inf\limits_{z_0 \in \mathbb{C}^n} \inf\{l_j(z)/l_j(z_0) \colon z \in \mathbb{D}^n[z_0, R/\mathbf{L}(z_0)]\}$, $\lambda_{2,j}(R)$ is defined analogously with replacement inf by sup.

For $F : \mathbb{C}^n \to \mathbb{C}^p$ let us introduce the sup-norm $|F(z)|_p = \max_{1 \le j \le p}\{|F_j(z)|\}$. The notation $A \le B$ for $A = (a_1, \dots, a_n)$, $B = (b_1, \dots, b_n) \in \mathbb{R}^n$ means that $a_j \le b_j$ for every $j \in \{1, \dots, n\}$. The following proposition was firstly deduced for analytic curves in [1]. Similar proposition was also obtained for analytic vector-valued functions $F \colon \mathbb{B}^2 \to \mathbb{C}^2$ in the unit ball $\mathbb{B}^2 = \{z \in \mathbb{C}^2 \colon |z_1|^2 + |z_2|^2 < 1\}$[2]. Here we present it for vector-valued entire functions $F \colon \mathbb{C}^n \to \mathbb{C}^p$.

Proposition 1. *Let $\mathbf{L} = (l_1(z), \dots, l_n(z))$ be a positive continuous function in $\mathbb{C}^n$. If each component f_s of an entire vector-valued function $F = (f_1, \dots, f_p) \colon \mathbb{C}^n \to \mathbb{C}^p$ is of bounded $\mathbf{L}$-index $N(\mathbf{L}, f_s)$ in joint variables then F is of bounded $\mathbf{L}$-index in joint variables in every norm, in particular, in the sup-norm and $N(\mathbf{L}; F) \le \max\{N(\mathbf{L}, f_s) \colon 1 \le s \le p\}$, and also F is of bounded $\mathbf{L}_*$-index in the Euclidean norm with $\mathbf{L}_*(z, w) \ge \sqrt{p}\mathbf{L}(z, w)$ and $N_E(\mathbf{L}_*, F) \le \max\{N(\mathbf{L}, f_s) : 1 \le s \le p\}$. (Here $N(\mathbf{L}, F)$ and $N_E(\mathbf{L}_*, F)$ are the $\mathbf{L}$-index and the $\mathbf{L}_*$-index in joint variables with the sup-norm and the Euclidean norm, respectively.)*

Theorem 2 ([3])**.** *Let $\mathbf{L} \in Q^n$. An entire vector-valued function $F \colon \mathbb{C}^n \to \mathbb{C}^p$ has bounded $\mathbf{L}$-index in joint variables if and only if for every $R \in \mathbb{R}^n_+$ there exist $n_0 \in \mathbb{Z}_+$, $p_0 > 0$ such that for all $z_0 \in \mathbb{C}^n$ there exists $K_0 \in \mathbb{Z}^n_+$, $\|K_0\| \le n_0$, satisfying inequality*

$$\max\Big\{\frac{|F^{(K)}(z)|_p}{K!\mathbf{L}^K(z)}:\|K\|\le n_0, z\in\mathbb{D}^n[z_0,R/\mathbf{L}(z_0)]\Big\}\le p_0\frac{|F^{(K_0)}(z_0)|_p}{K_0!\mathbf{L}^{K_0}(z_0)}.$$

This theorem is basic in the theory of functions of bounded index. Theorem 2 implies also the following corollary.

Corollary 3. *Let $\mathbf{L}\in Q^n$. An entire vector-function $F\colon\mathbb{C}^n\to\mathbb{C}^p$ has bounded $\mathbf{L}$-index in joint variables in the* sup-*norm if and only if it has bounded $\mathbf{L}$-index in joint variables in the norm $\|\cdot\|_0$.*

Theorem 4. *Let $\mathbf{L}\in Q^n$. In order that an entire vector-valued function $F\colon\mathbb{C}^n\to\mathbb{C}^p$ be of bounded $\mathbf{L}$-index in joint variables it is necessary that for all $R\in\mathbb{R}^n_+$ there exist $n_0\in\mathbb{Z}_+$, $p_1\ge1$ such that for all $z_0\in\mathbb{C}^n$ there exists $K_0\in\mathbb{Z}^n_+$, $\|K_0\|\le n_0$, satisfying inequality*

$$\max\{|F^{(K_0)}(z)|_p: z\in\mathbb{D}^n[z_0,R/\mathbf{L}(z_0)]\}\le p_1|F^{(K_0)}(z_0)|_p \tag{1}$$

and it is sufficiently that for all $R\in\mathbb{R}^n_+$ there exist $n_0\in\mathbb{Z}_+$, $p_1\ge1$ $\forall z_0\in\mathbb{C}^n$ $\exists K_1^0=(k_1^0,0,\ldots,0)$, $\exists K_2^0=(0,k_2^0,0,\ldots,0),\ldots,\exists K_n^0=(0,\ldots,0,k_n^0)$: $k_j^0\le n_0$, and

$$(\forall j\in\{1,\ldots,n\}):\ \max\{|F^{(K_j^0)}(z)|_p: z\in\mathbb{D}^n[z_0,R/\mathbf{L}(z_0)]\}\le p_1|F^{(K_j^0)}(z_0)|_p \tag{2}$$

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Separable cubic stochastic operators

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Simplex, Cubic stochastic operator, Separable cubic stochastic operator, Identity matrix, Skew symmetric matrix.

A cubic stochastic operator (CSO) has meaning of a population evolution operator, which arises as follows: Consider a population consisting of m species.

Let $x^{(0)} = (x_1^{(0)}, ..., x_m^{(0)})$ be the probability distribution of species in the initial generations, and $P_{ijk,l}$ the probability that individuals in the ith, jth and kth species interbreed to produce an individual l. Then the probability distribution $x' = (x'_1, ..., x'_m)$ of the species in the first generation can be found by the total probability i.e.

$$W : x'_l = \sum_{i,j,k=1}^{m} P_{ijk,l} x_i^0 x_j^0 x_k^0, \quad l \in E = \{1, ..., m\},$$

where the a matrix $\mathbf{P} \equiv \mathbf{P}(W) = \{P_{ijk,l}\}_{ijk,l=1}^{m}$ satisfying the following properties

$$P_{ijk,l} = P_{kij,l} = P_{ikj,l} = P_{kji,l} = P_{jik,l} = P_{jki,l} \geq 0, \ \sum_{l=1}^{m} P_{ijk,l} = 1 \quad for\ each \quad i, j, k \in E. \tag{1}$$

We define a map W of the simplex

$$S^{m-1} = \left\{ x = (x_1, ..., x_m) \in R^m : x_i \geq 0, \ \sum_{i=1}^{m} x_i = 1 \right\},$$

into itself, by the following rule

$$W : x'_l = \sum_{i,j,k=1}^{m} P_{ijk,l} x_i x_j x_k, \quad l \in E. \tag{2}$$

Definition 1. The operator W (2) is called cubic stochastic operator (CSO).

In this paper we consider CSO (2), (1) with additional properties

$$P_{ijk,l} = a_{il} b_{jl} c_{kl}, \quad for\ all \quad i, j, k, l \in E, \tag{3}$$

where $a_{il}, b_{jl}, c_{kl} \in R$ entries of quadratic matrices $A = (a_{il})$, $B = (b_{jl})$ and $C = (c_{kl})$ such that the properties (1) are satisfied for the coefficients (3).

Then the CSO W corresponding to the matrices A, B and C has the form

$$x'_l = (W(x))_l = (A(x))_l (B(x))_l (C(x))_l, \ \ for\ all \ \ l \in E, \tag{4}$$

where

$$(A(x))_l = \sum_{i=1}^{m} a_{il} x_i, (B(x))_l = \sum_{j=1}^{m} b_{jl} x_j, (C(x))_l = \sum_{k=1}^{m} c_{kl} x_k. \tag{5}$$

Definition 2. The CSO (4) is called separable cubic stochastic operator (SCSO) and we denote it by $W = (A, B, C)$.

We denote by $\mathbf{m}$ quadratic matrix $m \times m$ with elements $m_{ij} = m$, $i, j \in E$

If $A = I_m$ be an identity $m \times m$ matrix, i.e. $a_{il} = 0$ for $i \neq l$ and $a_{ii} = 1$ for all $i, l \in E$, in properties (5). Then the following simple Proposition is useful.

Proposition 3. *Let $A = I_m$, then for matrices $B = (b_{jl})_{j,l=1}^m$ and $C = (c_{kl})_{k,l=1}^m$ of SCSO $W = (I_m, B, C)$ the following property is true: $b_{jl}c_{kl} \geq 0$, $BC^T = \mathbf{m}$ where and C^T is the transpose of C.*

Proposition 4. *If $A = I_3$, $B = (b_{jl})_{j,l=1}^3$ is a skew symmetric matrix. The following equation solvable*

$$B\left(c^{(k)}\right)^T = (3,3,3), \quad k = 1,2,3 \tag{6}$$

if and only if $b_{23} = b_{13} - b_{12}$. Moreover, for the solution $C = (c_{kl})_{k,l=1}^3$ is the following equality

$$\left(c^{(k)}\right)^T = \left(c_{1k}, \frac{3 + b_{13}c_{1k}}{b_{12} - b_{13}}, \frac{3 + b_{12}c_{1k}}{b_{13} - b_{12}}\right), \quad k = 1,2,3 \tag{7}$$

is true, where $(c^{(k)})$ is a row of matrix $C = (c_{kl})_{k,l=1}^3$.

Theorem 5. *If $A = I_3$, $B = (b_{jl})_{j,l=1}^3$ is a skew symmetric matrix and equality (6) is hold, then the SCSO is the quadratic stochastic operator.*

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Asymptotically equivalent subspaces of metric spaces

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We investigate the asymptotic behavior of unbounded metric spaces at infinity. To do this we consider a sequence of rescaling metric spaces $\left(X, \frac{1}{r_n}d\right)$ generated by a metric space (X,d) and a scaling sequence $(r_n)_{n\in\mathbb{N}}$ of positive reals with $r_n \to \infty$. By definition, the pretangent spaces to (X,d) at infinity $\Omega^X_{\infty,\tilde{r}}$ are limit points of this rescaling sequence. We found the necessary and sufficient conditions under which two given unbounded subspaces of (X,d) have the same pretangent spaces at infinity.

Definition 1. Let (X,d) be an unbounded metric space. Two sequences $\tilde{x} = (x_n)_{n\in\mathbb{N}} \subset X$ and $\tilde{y} = (y_n)_{n\in\mathbb{N}} \subset X$ are *mutually stable* with respect to a scaling sequence $\tilde{r} = (r_n)_{n\in\mathbb{N}}$ if there is a finite limit

$$\lim_{n\to\infty} \frac{d(x_n, y_n)}{r_n}.$$

For every unbounded metric space (X,d) and every scaling sequence $\tilde{r}$, we denote by $Seq(X,\tilde{r})$ the set of all sequences $\tilde{x} = (x_n)_{n\in\mathbb{N}} \subset X$ for which $\lim\limits_{n\to\infty} d(x_n,p) = \infty$ and there is a finite limit

$$\lim_{n\to\infty} \frac{d(x_n, p)}{r_n},$$

where p is a fixed point of X.

Definition 2. A set $F \subseteq Seq(X,\tilde{r})$ is *self-stable* if any two $\tilde{x}, \tilde{y} \in F$ are mutually stable. F is *maximal self-stable* if it is self-stable and, for arbitrary $\tilde{y} \in Seq(X,\tilde{r})$, we have either $\tilde{y} \in F$ or there is $\tilde{x} \in F$ such that $\tilde{x}$ and $\tilde{y}$ are not mutually stable.

Let (X,d) be an unbounded metric space, let Y and Z be unbounded subspaces of X and let $\tilde{r} = (r_n)_{n\in\mathbb{N}}$ be a scaling sequence.

Definition 3. The subspaces Y and Z are *asymptotically equivalent* with respect to $\tilde{r}$ if for every

$$\tilde{y}_1 = (y_n^{(1)})_{n\in\mathbb{N}} \in Seq(Y,\tilde{r}) \quad \text{and} \quad \tilde{z}_1 = (z_n^{(1)})_{n\in\mathbb{N}} \in Seq(Z,\tilde{r})$$

there exist

$$\tilde{y}_2 = (y_n^{(2)})_{n\in\mathbb{N}} \in Seq(Y,\tilde{r}) \quad \text{and} \quad \tilde{z}_2 = (z_n^{(2)})_{n\in\mathbb{N}} \in Seq(Z,\tilde{r})$$

such that

$$\lim_{n\to\infty} \frac{d(y_n^{(1)}, z_n^{(2)})}{r_n} = \lim_{n\to\infty} \frac{d(y_n^{(2)}, z_n^{(1)})}{r_n} = 0.$$

We shall say that Y and Z are *strongly asymptotically equivalent* if Y and Z are asymptotically equivalent for all scaling sequences $\tilde{r}$.

Let (X,d) be a metric space and let $p \in X$. For every $t > 0$ we denote by $S(p,t)$ the sphere with the radius t and the center p,

$$S(p,t) := \{x \in X \colon d(x,p) = t\},$$

and for every $Y \subseteq X$ we write

$$S_t^Y := S(p,t) \cap Y.$$

Let Y and Z be subspaces of (X,d). Define

$$\varepsilon(t, Z, Y) := \sup_{z \in S_t^Z} \inf_{y \in Y} d(z,y)$$

and

$$\varepsilon(t) = \max\{\varepsilon(t,Z,Y), \varepsilon(t,Y,Z)\},$$

where we set $\varepsilon(t,Z,Y) = 0$ if $S_t^Z = \varnothing$ and, respectively, $\varepsilon(t,Y,Z) = 0$ if $S_t^Y = \varnothing$.

Theorem 4. *Let Y and Z be unbounded subspaces of a metric space (X,d). Then Y and Z are strongly asymptotically equivalent if and only if*

$$\lim_{t\to\infty} \frac{\varepsilon(t)}{t} = 0.$$

Corollary 5. *Let (X,d) be an unbounded metric space and let Y be an unbounded subspace of X. Then the following conditions are equivalent.*

(1) *For every $\tilde{r}$ and every maximal self-stable $\tilde{X}_{\infty,\tilde{r}} \subseteq Seq(X,\tilde{r})$ there is a maximal self-stable $\tilde{Y}_{\infty,\tilde{r}} \subseteq Seq(X,\tilde{r})$ such that $\tilde{Y}_{\infty,\tilde{r}} \subseteq \tilde{X}_{\infty,\tilde{r}}$ and the embedding $E_{m_Y} : \Omega^Y_{\infty,\tilde{r}} \to \Omega^X_{\infty,\tilde{r}}$ is an isometry.*

(2) *The equality*

$$\lim_{t\to\infty} \frac{\varepsilon(t,X,Y)}{t} = 0$$

holds.

(3) *X and Y are strongly asymptotically equivalent.*

Remark 6. Theorem 4 and Corollary 5 can be considered as asymptotic variants of previously proved facts from [1].

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Isomorphic issues about the CTCs in Quantum Physics

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The main solutions to the Polchinski's paradox [7] are Novikov's self-consistent causal loops [5] referring to a Reciprocity Principle (RP) in physics ([2, 3, 4]) whereby the past determines the future as well as the future determines the past. The recent proposal of a quantum circuit formulation of the famous wormhole billiard ball paradox [1] has renovated the interest for closed time-like curves (CTCs) applied to elementary particles. We wish to enrich such discussion by focusing on an electron entering in a time-travel tunnel so that it can collide with its past self at low energy. We investigate the graph isomorphism (GI) of two alternative cases about the exiting particle: 1) If it is still an *electron*, then the collision deflects the trajectory of the incoming particle just towards the tunnel entrance (within a stable time loop). 2) If it is a *positron*, i.e., matter going backwards in time [6], then the interaction with the incoming electron is a process of pair production which is reversed inside the tunnel (as annihilation) according to the RP. Our GI analysis raises open questions ranging from the role of a preferential arrow of time to the validity of the law of inertia in chronology violations.

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Geometrical Langlands Ramifications and Differential Operators Classification by Verma Module Extensions

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Studies realized to the differential operator classification have been realized using the generalized Verma modules as classifying spaces defined by the geometrical Langlands correspondences through of functors characterized for integral transforms to define the equivalences between geometrical objects of holomorphic bundles and objects of an algebra of operators. Likewise are characterized the Lie algebras of these differential operators under the Hecke categories and their classifying spaces as Verma modules extensions. Likewise, is had the following result:

Theorem 1. (F.Bulnes). *The derived category of quasi-G- invariants* $D_{G/H}-$ *modules formed with the extended and generalized Verma modules given for* ${}^{L}\Phi^{\mu}({}^{L}(\mathcal{M})) = \mathcal{M} \boxtimes \rho^{\mu}(\mathbb{V}), \forall \mathbb{V} \in (Loc_L)$*, can be identified for a critically twisted sheaves category of D-modules on the moduli stack* $\mathrm{Bun}_{G,y}, \forall y \in X$ *(singularity) identified by the Hecke category* $\mathcal{H}_{G,K,y}$*, (geometrical Langlands correspondence), if this is an image of integral transforms acting on ramifications of the Hecke category* $\mathcal{H}_G, \forall \lambda \in \mathrm{h}*$*(for example* $\mathcal{H}_{G,\lambda}$*) on the flag manifold* G/B*, with weight corresponding to twisted differential operators on* $\mathrm{Bun}_{G,y}$.

Key words: Langlands correspondence, Hecke sheaves category, moduli stacks, Verma modules, generalized D-modules, Verma Module Extensions.

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Conharmonic Transformations of Locally Conformal Kähler Manifolds

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A Hermitian manifold (M^{2m}, J, g) is called a *locally conformal Kähler manifold (LCK - manifold)* if there is an open cover $\mathfrak{U} = \{U_\alpha\}_{\alpha\in A}$ of M^{2m} and a family $\{\sigma_\alpha\}_{\alpha\in A}$ of C^∞ functions $\sigma_\alpha : U_\alpha \to \mathbb{R}$ so that each local metric

$$\hat{g}_\alpha = e^{-2\sigma_\alpha} g|_{U_\alpha}$$

is Kählerian. An LCK - manifold is endowed with some form ω, so called *Lee form* which can be calculated as [1]

$$\omega = \frac{1}{m-1}\delta\Omega \circ J.$$

The form should be closed:

$$d\omega = 0.$$

Here and below, we denote by comma covariant differentiation with respect to the Levi-Civita connection of (M^{2m}, J, g).

If a contravariant analitic vector field ξ generates conformal infinitesimal transformation of an LCK-manifold, then the field satisfy the system [2]

$$\begin{array}{ll} 1) & \xi_{i,j} = \xi_{ij}; \\ 2) & \xi_{i,j} + \xi_{j,i} = \left(\omega_\alpha \xi^\alpha + C\right) g_{ij}; \\ 3) & \xi_{i,jk} = \xi_\alpha R^\alpha_{kji} + \frac{1}{2}\left(\left(\omega_\alpha\xi^\alpha\right)_{,k} g_{ij} + \left(\omega_\alpha\xi^\alpha\right)_{,j} g_{ik} - \left(\omega_\alpha\xi^\alpha\right)_{,i} g_{jk}\right); \\ 4) & J^i_{j,k}\xi^k - J^\alpha_j \xi^i_{,\alpha} + J^i_\alpha \xi^\alpha_{,j} = 0. \end{array} \tag{1}$$

If a conformal transformation (101) also preserves a product Rg_{ij}, i. e. the equation

$$\mathfrak{L}_\xi\left(Rg_{ij}\right) = 0 \tag{2}$$

holds, then the transformation is called *conharmonic.* We obtain the theorem.

Theorem 1. *If an LCK-manifold (M^{2m}, J, g) of non-zero scalar curvature admits nontrivial conharmonic transformations, then the general solution of the PDE system (101)-(2) depends on no more than $m^2 + 2m$ essential parameters.*

Also we have proved that the tensor

$$P_{ij} \stackrel{\text{def}}{=} \frac{1}{n-2}R_{ij} - \frac{1}{2}\omega_{i,j} - \frac{1}{4}\omega_i\omega_j + \frac{1}{8}\omega^{\alpha}\omega_{\alpha}g_{ij}$$

is preserved by conharmonic transformations.

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Applications of Linking to the Study of Causality

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We will discuss the results about causality in spacetimes and Legendrian linking. The spheres are linked in the space of all light rays associated to the spacetimes. The results were obtained in the joint works with Stefan Nemirovski and, in particular, solve the Low conjecture, the Legendrian Low conjecture of Natario and Tod and the problem communicated by Penrose on Arnold problem list.

Problem on extremal decomposition of the complex plane

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Let $\mathbb{N}$, $\mathbb{R}$ be the sets of natural and real numbers, respectively, let $\mathbb{C}$ be the complex plane, and let $\overline{\mathbb{C}} = \mathbb{C}\bigcup\{\infty\}$ be its one-point compactification, $\mathbb{R}^+ = (0,\infty)$. Let $r(B,a)$ be the inner radius of the domain $B \subset \overline{\mathbb{C}}$ relative to a point $a \in B$. The inner radius of the domain B is connected with Green's generalized function $g_B(z,a)$ of the domain B by the relations

$$g_B(z,a) = -\ln|z-a| + \ln r(B,a) + o(1), \quad z \to a,$$

$$g_B(z,\infty) = \ln|z| + \ln r(B,\infty) + o(1), \quad z \to \infty.$$

Definition 1. Let $n \in \mathbb{N}$, $n \geqslant 2$. The system of points $A_n := \left\{a_k \in \mathbb{C} : \ k = \overline{1,n}\right\}$ is called n-ray, if $|a_k| \in \mathbb{R}^+$ for $k = \overline{1,n}$ and $0 = \arg a_1 < \arg a_2 < \ldots < \arg a_n < 2\pi$.

Denote $\alpha_k := \dfrac{1}{\pi}\arg\dfrac{a_{k+1}}{a_k}$, $\alpha_{n+1} := \alpha_1$, $k = \overline{1,n}$, $\sum\limits_{k=1}^{n} \alpha_k = 2$.

Problem 2. (V.N. Dubinin [1, 2]) For all values of the parameter $\gamma \in (0,n]$ to show that the maximum of the functional

$$I_n(\gamma) = r^{\gamma}\left(B_0,0\right)\prod_{k=1}^{n} r\left(B_k,a_k\right),$$

where B_0, B_1, B_2,..., B_n, $n \geqslant 2$, are pairwise disjoint domains in $\overline{\mathbb{C}}$, $a_0 = 0$, $|a_k| = 1$, $k = \overline{1,n}$, is attained for the configuration of domains B_k and points a_k which possesses the n-fold symmetry.

In work [1], the above-formulated problem was solved for the value of the parameter $\gamma = 1$ and all values of the natural parameter $n \geqslant 2$. Namely, it was shown that the following inequality holds

$$r(B_0,0)\prod_{k=1}^{n} r(B_k,a_k) \leqslant r\left(D_0,0\right)\prod_{k=1}^{n} r\left(D_k,d_k\right),$$

where d_k, D_k, $k = \overline{0,n}$, are the poles and circular domains of the quadratic differential

$$Q(w)dw^2 = -\frac{(n^2-1)w^n+1}{w^2(w^n-1)^2}\,dw^2.$$

In work [3], L.V. Kovalev got its solution for definite sufficiently strict limitations on the geometry of arrangement of the systems of points on a unit circle, namely, for systems of points for which the following inequalities hold

$$0 < \alpha_k \leqslant 2/\sqrt{\gamma}, \quad k = \overline{1,n}, \quad n \geqslant 5.$$

In work [4], it was shown that the result by L.V. Kovalev is true for $n = 4$. The solution of this problem for $\gamma \in (0,1]$ was given in work [5]. Some partial cases of this problem were studied, for example, in [6–10].

For the further analysis, we calculate the quantity

$$I_n^0(\gamma) = r^{\gamma}\left(D_0,0\right)\prod_{k=1}^{n} r\left(D_k,d_k\right),$$

where d_k, D_k, $k=\overline{0,n}$, $d_0=0$, are, respectively, the poles and circular domains of the quadratic differential

$$Q(w)dw^2 = -\frac{(n^2-\gamma)w^n+\gamma}{w^2(w^n-1)^2}\,dw^2.$$

As was shown in [1, 2, 3, 6], the quantity $I_n^0(\gamma)$ takes the form

$$I_n^0(\gamma)=\left(\frac{4}{n}\right)^n \frac{\left(\frac{4\gamma}{n^2}\right)^{\frac{\gamma}{n}}}{\left(1-\frac{\gamma}{n^2}\right)^{n+\frac{\gamma}{n}}} \left(\frac{1-\frac{\sqrt{\gamma}}{n}}{1+\frac{\sqrt{\gamma}}{n}}\right)^{2\sqrt{\gamma}}.$$

Theorem 3. [9] *Let $\gamma\in(1,2]$. Then, for any different points a_1 and a_2 of a unit circle and any mutually disjoint domains B_0, B_1, B_2, $a_1\in B_1\subset\overline{\mathbb{C}}$, $a_2\in B_2\subset\overline{\mathbb{C}}$, $a_0=0\in B_0\subset\overline{\mathbb{C}}$, the inequality*

$$r^{\gamma}\left(B_0,0\right) r\left(B_1,a_1\right) r\left(B_2,a_2\right)\leqslant I_2^0(\gamma)\left(\frac{1}{2}\left|a_1-a_2\right|\right)^{2-\gamma}.$$

is true. The sign of equality in this inequality is attained, when the points a_0, a_1, a_2 and the domains B_0, B_1, B_2 are, respectively, the poles and circular domains of the quadratic differential

$$Q(w)dw^2=-\frac{(4-\gamma)w^2+\gamma}{w^2(w^2-1)^2}dw^2.$$

Remark 4. Theorem 3 yields the complete solution of the above-posed problem of finding the maximum of product of inner radii of two domains relative to the points of a unit circle on the degree γ of the inner radius of the domain relative to the origin at arbitrary $\gamma\in(0,2]$, provided that all three domains are mutually non-overlapping domains.

Theorem 5. [9] *Let $n\in\mathbb{N}$, $n\geqslant 3$, $\gamma\in(1,n]$. Then, for any system of different points $A_n=\{a_k\}_{k=1}^n\in\mathbb{C}\backslash\{0\}$ of a unit circle and for any collection of mutually disjoint domains B_0, B_k, $a_0=0\in B_0\subset\overline{\mathbb{C}}$, $a_k\in B_k\subset\overline{\mathbb{C}}$, $k=\overline{1,n}$, the following inequality holds*

$$r^{\gamma}(B_0,0)\prod_{k=1}^{n} r(B_k,a_k)\leqslant\left(\sin\frac{\pi}{n}\right)^{n-\gamma}\left(I_2^0\left(\frac{2\gamma}{n}\right)\right)^{\frac{n}{2}}.$$

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Parallel spinors on Lorentzian Weyl spaces

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In that talk I will present the recent results of joint work with Anton S. Galaev on Lorentzian Weyl spin manifolds admitting weighted parallel spinors [2].

Parallel spinors are special Killing spinors which represent supersymmetry generators of supersymmetric field theories and supergravity theories. The physical motivation for study Weyl spaces with weighted parallel spinors may be found in [3]. The work [3] provides a deep investigation of weighted parallel spinors on Einstein-Weyl manifolds of Lorentzian signature with a special attention to the dimensions 4 and 6. The techniques developed for classification of supergravity solutions was used in that work.

We provide a description of simply connected Lorentzian Weyl spin manifolds admitting weighted parallel spinors. The main tool for that are holonomy groups. There is a one-to-one correspondence between the parallel spinors and the holonomy-invariant elements of the spinor module. This correspondence is used in known results describing the following simply connected spin manifolds with parallel spinors: Riemannian manifolds, pseudo-Riemannian manifolds with irreducible holonomy groups, Lorentzian manifolds.

Using the recent classification of holonomy algebras of Lorentzian Weyl manifolds [1], we classify the holonomy algebras of Lorentzian Weyl spaces admitting weighted parallel spinors. It turns out that for non-closed Weyl structures, there are two types of such algebras. In each case, the dimension of the space of parallel spinors is found.

For Lorentzian Weyl manifolds admitting recurrent null vector fields are introduced special local coordinates similar to Kundt and Walker ones. Using that, the local form of all Lorentzian Weyl spin manifolds with weighted parallel spinors is given. The Einstein-Weyl equation for the obtained Weyl structures is analyzed and examples of Einstein-Weyl spaces with weighted parallel spinors are given. Some examples have previously appeared in [3] and other literature. It turns out that the Einstein-Weyl equation implies that the weight of a non-zero weighted parallel spinor is equal to $\dim M - 4$. Parallel spinors of that weight were studied in [3]. In contrast, we describe Weyl structures with non-zero weighted parallel spinors of arbitrary weight.

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Matrix problems, triangulated categories and stable homotopy types

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The talk is a survey of some results on classifications of stable homotopy types of polyhedra (finite CW-complexes). We present technical tools for calculations in triangulates categories, which are related to *marix problems*, namely, to *bimodule categories*. Applying this technique to the stable homotopy category [1] we obtain a complete classification of stable homotopy types of polyhedra having cells at most in 4 successive dimensions and of torsion free polyhedra having cells at most in 7 successive dimensions. For details, see [2, 3]. These results were mainly obtained in collaboration with H.-J. Baues.

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On the properties smooth manifolds defined by intersections

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The report is devoted to study properties and construction an examples of the $(3,4)$-dim smooth manifolds contained the surfaces of constant curvature.

At the first will be considered the Dyck-surface (W.Dyck,1888) defined by the algebraic equation

$$\left({z_1}^2+{z_2}^2\right)\left({z_1}^2+{z_2}^2+{z_3}^2\right)-z_3\left(4\,{z_1}^2+2\,{z_2}^2\right)=0, \tag{1}$$

where z_1, z_2, z_3 are the complex coordinates: $z_1 = x + Ia$, $z_2 = y + Ib$, $z_3 = z + Ic$, $I^2 = -1$. The complex surface (1) is generalization of real projective surface and belongs to the class of the one-side surfaces having an important applications in various branch of modern algebraic topology (J.Milnor,1968).

Proposition 1. *Joint consideration both equations (1) and the equation of the 5D-sphere $|z_1|^2 + |z_2|^2+|z_3|^2 = 1$ with real coordinates (x, y, z, a, b, c), in general, lead us to the some 4D-space, where (t)-is auxiliary parameter:*

$$^4G(a,b,c,z,t) = bc^2 - b + b^3 + bz^2 + 2\,ct - ztc - bt^2 + ba^2 = 0, \tag{2}$$

containing 3D-subspace with the equation

$$^3H(a,b,c,z) = 4\,a^2b^2 - 4\,b^2 + 4\,b^4 + 4\,c^2 + 4\,b^2c^2 - 4\,zc^2 + 4\,b^2z^2 + z^2c^2 = 0. \tag{3}$$

With the equation (3) at the condition $b = b(a)$ can be associated an invariant second order ODE of the form $b'' - A_1b'^3 - 3A_2b'^2 - 3A_3b' - A_4 = 0$, $A_i = A_i(a,b)$ having the General integral

$$^3F(a,b,C_1,C_2) = -\,C_1 - \,b^4 - \,a^2b^2 + 2\,b^2C_2 = 0.$$

having an algebraic curve of genus $g_{a,b} = 1$, and which is placed on the 2D-surface, equipped by the metrics of const positive curvature $K = 1$

$$\phi^2ds^2 = \psi_1(x,y)dx^2+2\psi_2(x,y)dx\,dy+\psi_3(x,y)dy^2, \quad \phi(x,y) = \psi_1(x,y)\psi_3(x,y)-(\psi_2(x,y))^2\,. \tag{4}$$

The components ψ_i of the metrics are determined from the system (M.R.Liouville,1897)

$$\psi_{1x} + 2\,A_3\psi_1 - 2\,A_4\psi_2 = 0, \quad \psi_{3y} + 2\,A_1\psi_2 - 2\,A_2\psi_3 = 0,$$

$$\psi_{1y} + 2\psi_{2x} - 2\,A_3\psi_2 + 4\,A_2\psi_1 - 2\,A_4\psi_3 = 0, \quad \psi_{3x} + 2\,\psi_{2y} + 2\,A_2\psi_2 - 4\,A_3\psi_3 + 2\,A_1\psi_1 = 0. \tag{5}$$

In the second part of report we consider some examples of the Brieskorn type manifolds which are the intersection of the fife-dimensional sphere with the singular manifold ($l = 2$, $m = 3$, $n = 5$)

$$|z_1|^2 + |z_2|^2 + |z_3|^2 = 1, \quad z_1^2 + z_2^3 + z_3^5 = 0. \tag{6}$$

Proposition 2. *From the equations of the system (6)*

$$x^2 + a^2 + y^2 + b^2 + z^2 + c^2 - 1 = 0, \quad 2\,xa + 3\,y^2b - b^3 + 5\,z^4c - 10\,z^2c^3 + c^5 = 0,$$

$$x^2 - a^2 + y^3 - 3\,yb^2 + z^5 - 10\,z^3c^2 + 5\,zc^4 = 0, \tag{7}$$

on the six real coordinates $z_1 = x + Ia$, $z_2 = y + Ib, z_3 = z + Ic$, in the case the relation $z = z(y)$ holds, the linearizable the second order ODE

$$\frac{d^2}{dy^2}z(y) = \frac{\left(\frac{d}{dy}z(y)\right)\left(-y\frac{d}{dy}z(y)+z(y)\right)}{yz(y)}$$

can be derived. It has general integral

$$F(z,y,C_1,C_2) = z - \sqrt{C_1\,y^2 + 2\,C_2} = 0,$$

containing the $2D$*-surface of constant curvature with the components of the metrics defined from* (1)*,*

$$\psi_2(x,y) = \frac{\left(C_3\,x^2 + C_4\right)y^{5/3}}{\sqrt[3]{x}} + \frac{C_1\,x^2 + C_2}{\sqrt[3]{x}\,\sqrt[3]{y}},\ \ \psi_1(x,y) = -\frac{x^{2/3}\left(-C_6 + C_3\,y^4 + 2\,C_1\,y^2\right)}{y^{4/3}},$$

$$\psi_3(x,y) = -\frac{y^{2/3}\left(-C_5 + x^4\,C_3 + 2\,x^2\,C_4\right)}{x^{4/3}}$$

with the parameters C_i*.*

By analogy are considered the case of tetrahedral space which corresponds to the intersection of the fife-dimensional sphere with singular manifold ($l=2,\quad m=3,\quad n=4$*)*

$$|z_1|^2 + |z_2|^2 + |z_3|^2 = 1, \quad z_1^2 + z_2^3 + z_3^4 = 0, \tag{8}$$

for which corresponding ODE has the form

$$\frac{d^2}{dy^2}z(y) = 1/2\,\frac{\left(\frac{d}{dy}z(y)\right)\left(3\,\left(\frac{d}{dy}z(y)\right)^2 y^2 - 6\,\left(\frac{d}{dy}z(y)\right)yz(y) + 2\,(z(y))^2\right)}{y\,(z(y))^2}$$

and the octahedral space defined by the condition

$$|z_1|^2 + |z_2|^2 + |z_3|^2 = 1, \quad z_1^2 + z_2^3 + z_2 z_3^3 = 0, \tag{9}$$

with the ODE: $\frac{d^2}{dy^2}z(y) = -\frac{\left(\frac{d}{dy}z(y)\right)^2}{z(y)}$*, and corresponding metrics* (4) *with the components* $\psi_i(x,y)$

$$\psi_3(x,y) = -2\,y^{2/3}C_3\,x^2 - 4\,y^{2/3}C_4\,x + C_5\,y^{2/3},\ \ \psi_2(x,y) = \frac{y^2\,C_3\,x + y^2\,C_4 + C_1\,x + C_2}{\sqrt[3]{y}},$$

$$\psi_1(x,y) = \frac{-1/2\,C_3\,y^4 - C_1\,y^2 + C_6}{y^{4/3}}.$$

To studying a moore detail properties of considered spaces can be used the $4D$-Riemann extensions

$$ds^2 = 2\,(zA_3 - tA_4)\,dx^2 + 4\,(zA_2 - tA_3)\,dx\,dy + 2\,(zA_1 - tA_2)\,dy^2 + 2\,dy\,dz + 2\,dy\,dt$$

of $2D$-metrics and with help of the Liouville-Tresse-Cartan invariants to investigated topological properties of the Brieskorn manifolds.

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Some applications of transversality for infinite dimensional manifolds

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We present some transversality results for a category of Fréchet manifolds, the so-called MC^k-Fréchet manifolds. In this context, we apply the obtained transversality results to construct the degree of nonlinear Fredholm mappings by virtue of which we prove a rank theorem, an invariance of domain theorem and a Bursuk-Ulam type theorem.

We refer to [1, 2] for the basic definitions and result regarding MC^k-Fréchet manifolds. We assume that E, F are Fréchet spaces and $\mathcal{U} \Subset E$ is an open subset, also that M, N are MC^k-Fréchet manifolds.

Theorem 1 (Transversality Theorem). *Let $\varphi : M \to N$ be an MC^k-mapping, $k \geq 1$, $S \subset N$ an MC^k-submanifold and $\varphi \pitchfork S$. Then, $\varphi^{-1}(S)$ is either empty of MC^k-submanifold of M with*

$$(T_x\varphi)^{-1}(T_yS) = T_x(\varphi^{-1}(S)),\ x \in \varphi^{-1}(S),\ y = \varphi(x).$$

If S has finite co-dimension in N, then $codim(\varphi^{-1}(S)) = codim S$. Moreover, if $\dim S = m < \infty$ and φ is an MC^k-Lipschitz-Fredholm mapping of index l, then $\dim \varphi^{-1}(S) = l + m$.

Theorem 2 (The Parametric Transversality Theorem). *Let A be a manifold of dimension n, $S \subset N$ a submanifold of finite co-dimension m. Let $\varphi : M \times A \to N$ be an MC^k-mapping, $k \geq \{1, n-m\}$. If φ is transversal to S, $\varphi \pitchfork S$, then the set of all points $x \in M$ such that the mappings*

$$\varphi_x : A \to N,\ (\varphi_x(\cdot) := \varphi(x, \cdot))$$

are transversal to S, is residual M.

Theorem 3 (Rank theorem for MC^k-mappings). *Let $\varphi : \mathcal{U} \Subset E \to F$ be an MC^k-mapping, $k \geq 1$. Suppose $u_0 \in \mathcal{U}$ and $D\varphi(u_0)$ has closed split image $\mathbf{F_1}$ with closed complement $\mathbf{F_2}$ and split kernel $\mathbf{E_2}$ with closed complement $\mathbf{E_1}$. Also, assume $D\varphi(\mathcal{U})(E)$ is closed in F and $D\varphi(u)|_{\mathbf{E_1}} : \mathbf{E_1} \to D\varphi(u)(E)$ is an MC^k-isomorphism for each $u \in \mathcal{U}$. Then, there exist open sets $\mathcal{U}_1 \Subset \mathbf{F_1} \oplus \mathbf{E_2}$, $\mathcal{U}_2 \Subset E$, $\mathcal{V}_1 \Subset F$, and $\mathcal{V}_2 \Subset F$ and there are MC^k-diffeomorphisms $\phi : \mathcal{V}_1 \to \mathcal{V}_2$ and $\psi : \mathcal{U}_1 \to \mathcal{U}_2$ such that*

$$(\phi \circ \varphi \circ \psi)(f, e) = (f, 0), \quad \forall (f, e) \in \mathcal{U}_1.$$

Theorem 4 (Invariance of domain for Lipschitz-Fredholm mappings). *Let $\varphi : M \to N$ be an MC^k-Lipschitz-Fredholm mapping of index zero, $k > 1$. If φ is locally injective, then φ is open.*

Definition 5. Let $\varphi : M \to N$ be a non-constant closed Lipschitz-Fredholm mapping with index $l \geq 0$ of class MC^ksuch that $k > l + 1$. We associate to φ a degree, denoted by $\deg \varphi$, defined as the non-oriented cobordism class of $\varphi^{-1}(q)$ for some regular value q. If $l = 0$, then $\deg \varphi \in \mathbb{Z}_2$ is the number modulo 2 of preimage of a regular value.

Theorem 6 (Bursuk-Ulam Theorem). *Let $\varphi : \overline{\mathcal{U}} \to F$ be a non-constant closed Lipschitz-Fredholom mapping of class MC^2 with index zero, where $U \Subset F$ is a centrally symmetric and bounded. If φ is odd and for $u \in \overline{U}$ we have $u \notin \varphi(\partial\overline{\mathcal{U}})$. Then $\deg(\varphi, 0_F) \equiv 1 \mod 2$.*

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Uniqueness theorems for almost periodic objects

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New uniqueness theorems are considered for various types of almost periodic objects: functions, measures, distributions, multisets, holomorphic and meromorphic functions, Fourier quasicrystals.

On symmetry reduction and some classes of invariant solutions of the $(1+3)$-dimensional homogeneous Monge-Ampère equation

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A solution of many problems of the geometry, theoretical and mathematical physics has reduced to the investigation of the Monge-Ampère equations in the spaces of different dimensions and different types.

It is well known that the symmetry reduction is one of the most powerful tools to investigate partial differential equations (PDEs) with non-trivial symmetry groups. In particular, for this purpose, we can use a classical Lie-Ovsiannikov method. This method, among the other, makes it possible to perform the symmetry reduction and construction of invariant solutions of those equations.

In 1984, Grundland, Harnad, and Winternitz pointed out that the reduced equations, obtained with the help of nonconjugate subalgebras of the same ranks of the Lie algebras of the symmetry groups of some PDEs, were of different types. They also investigated the similar phenomenon. The results obtained cannot be explained using the classical Lie-Ovsiannikov approach.

To try to explain some of the differences in the properties of the reduced equations for PDEs with nontrivial symmetry groups, we suggested to investigate the relationship between the structural properties of nonconjugate subalgebras of the same rank of the Lie algebras of the symmetry groups of those PDEs and the properties of the reduced equations corresponding with them.

At the present time, we have investigated the relationship between structural properties of the three-dimensional nonconjugate subalgebras of the same rank of the Lie algebra of the Poincaré group $P(1,4)$ and the properties of reduced equations for the $(1+3)$-dimensional homogeneous Monge-Ampére equation. We obtained the following types of the reduced equations:
- identities,
- the linear ordinary differential equations,
- the nonlinear ordinary differential equations,
- the partial differential equations.

Some classes of invariant solutions have been constructed.

In my report, I plan to present some of the results obtained concerning with reduction of the $(1+3)$-dimensional homogeneous Monge-Ampére equation to identities.

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Deformations of circle-valued Morse functions on 2-torus

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Let M be a smooth compact surface, X be a closed (possible empty) subset of M. By P we also denote either $\mathbb{R}$ or S^1. The group $\mathcal{D}(M,X)$ of diffeomorphisms of M fixed on X acts from the right on the space of smooth maps $C^\infty(M,P)$ by the rule

$$\gamma : C^\infty(M,P) \times \mathcal{D}(M,X) \to C^\infty(M,P), \qquad \gamma(f,h) = f \circ h.$$

With respect to γ we denote by

$$\mathcal{S}(f,X) = \{h \in \mathcal{D}(M,X) \,|\, f \circ h = f\},$$
$$\mathcal{O}(f,X) = \{f \circ h \,|h \in \mathcal{D}(M,X)\}$$

the *stabilizer* and the *orbit* of $f \in C^\infty(M,P)$. Endow strong Whitney C^∞-topologies on $C^\infty(M,P)$ and $\mathcal{D}(M,X)$; then for a map $f \in C^\infty(M,P)$ these topologies induce some topologies on $\mathcal{S}(f,X)$ and $\mathcal{O}(f,X)$. We denote by $\mathcal{D}_{\mathrm{id}}(M,X)$ a connected component of the identity map $\mathcal{D}(M,X)$, and by $\mathcal{O}_f(f,X)$ a connected component of $\mathcal{O}(f,X)$ containing f. If $X = \varnothing$ we omit the symbol "$\varnothing$" from our notation.

To state our main result we need a notion of wreath product of groups of a special kind. Let G be a group, $n \geq 1$ be an integer. A semi-direct product $G^n \rtimes \mathbb{Z}$ with respect to a non-effective $\mathbb{Z}$-action α on G^n by cyclic shifts

$$\alpha(b_0, b_1, \ldots, b_{n-1}; k) = (b_k, b_{1+k}, \ldots, b_{n+k-1}),$$

where all indexes are taken modulo n, will be denoted by $G \wr_n \mathbb{Z}$ and called a *wreath product* of G with $\mathbb{Z}$ under n.

The following theorem is our main result.

Theorem 1 ([1]). *Let f be a function from $\mathcal{F}(T^2,P)$ with at least one critical point and whose Kronrod-Reeb graph contains a cycle. Then there exist a cylinder $Q \subset T^2$ such that $f|_Q : Q \to P$ is a Morse function, $n \in \mathbb{N}$ such that there is an isomorphism*

$$\pi_1 \mathcal{O}_f(f) \cong \pi_0 \mathcal{S}'(f|_Q, \partial Q) \wr_n \mathbb{Z},$$

where $\mathcal{S}'(f|_Q, \partial Q) = \mathcal{S}(f|_Q, \partial Q) \cap \mathcal{D}_{\mathrm{id}}(Q, \partial Q)$.

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Realization of a graph as the Reeb graph of a Morse, Morse–Bott or round function

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Reeb graph R_f of a function $f : M \to \mathbb{R}$ is a topological space obtained by contracting the connected components of the level sets of f to points, endowed with the quotient topology; for a smooth function, connected components containing critical points are called *vertices*, i.e., the *Reeb graph of a smooth function* is the quotient space with marked points.

By a *graph* we understand a pseudograph (allowing loop edges and multiple edges); it has a geometric realization as a one-dimensional CW complex, in which 0-cells correspond to vertices and 1-cells to edges. A graph needs not to be connected.

Definition 1. We say that a Reeb graph R_f *has the structure of a finite graph* G, or R_f *is isomorphic to* G, or R_f *is* G, if there exists a homeomorphism $R_f \to G$ mapping one-to-one the vertices of R_f to the vertices of G.

Generally, the Reeb graph is not a finite graph; in our talk we consider a simple counterexamle. Recently Saeki proved a criterion:

Theorem 2 ([1]). *Let M be a closed manifold, $f : M \to \mathbb{R}$ a smooth function. Then the Reeb graph R_f has the structure of a finite graph if and only if f has a finite number of critical values.*

Every graph without loop edges is the Reeb graph of some function:

Theorem 3 ([2]). *Let G be a finite graph. Then there exist a closed manifold M, and a smooth function $f : M \to \mathbb{R}$ such that its Reeb graph R_f has the structure of G if and only if G has no loop edges.*

The problem of whether a finite graph is the Reeb graph of some function was first studied in 2006 by Sharko [3]. He considered functions with finite critical set $Crit(f)$. In particular, he showed that the graph shown in Figure 3.1 is not the Reeb graph of any such function.

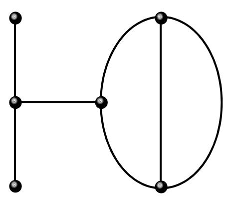

FIGURE 3.1.

Below we give criteria for a graph to be the Reeb graph of a function of a given class on a closed manifold: Morse, Morse–Bott, round, and in general smooth functions whose critical set $Crit(f)$ consists of a finite number of submanifolds.

In contrast to works of Michalak [4] and Martínez-Alfaro *et al.* [5] who studied the realization problem in terms of the graph orientation, the following criteria are given in terms of the graph structure, namely, the structure of its *leaf blocks*, i.e., maximal biconnected subgraphs containing at most one cut vertex:

Theorem 4 ([6]). *A graph G is isomorphic to the Reeb graph R_f of some smooth function f with finite $Crit(f)$ on a closed manifold if and only if G is finite, has no loop edges, and all its leaf blocks are path graphs on 2 vertices (closed intervals). The function f can be chosen Morse.*

Theorem 5 ([7]). *For any given $n \geq 2$, a graph G is the Reeb graph R_f of some smooth f whose $Crit(f)$ is a finite number of submanifolds, on closed n-manifold if and only if G is finite, has no loop edges, and each leaf block L has a vertex v with $\deg v \leq 2$, or two such vertices if L is a non-trivial (has an edge) connected component of G. The function f can be chosen Morse–Bott.*

This theorem shows that Sharko's graph in Figure 3.1 cannot be realized even as the Reeb graph of a function whose $Crit(f)$ is a finite number of submanifolds. Indeed, this graph has three leaf blocks, two of them being closed intervals, and the third leaf block has only 3-vertices.

Morse–Bott functions play a special role in the Reeb graph theory (cf. Theorem 3):

Theorem 6 ([8]). *Any finite graph is homeomorphic to the Reeb graph of a Morse–Bott function.*

Note that, in contrast to Theorem 3, this theorem is true even for graphs with loop edges.

Critical set of a *round function* consists of a finite number of circles. For a round function, the structure of its Reeb graph depends not only on leaf blocks, but also on the dimension of the manifold and its orientability:

Theorem 7 ([7]). *A graph G is isomorphic to the Reeb graph of a round function $f : M^n \to \mathbb{R}$ on a closed n-dimensional manifold if and only if G is finite, has no loop edges, and*

$$\textit{each its leaf block} \begin{cases} \textit{has a non-cut vertex } v \textit{ with } \deg v = 2 & \textit{if } n = 2, \textit{ orientable surface} \\ \textit{has a non-cut vertex } v \textit{ with } \deg v \leq 2 & \textit{if } n = 2, \textit{ non-orientable surface} \\ \textit{is a path graph on 2 vertices (closed interval)} & \textit{if } n \geq 3. \end{cases}$$

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Foliations in moduli spaces of abelian varieties and bounded global G-shtukas

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In this communication we review and study moduli spaces of abelian varieties, of p-divisible groups, of bounded global G-shtukas and their possible foliations in characteristic p. At first I recall the definitions of the above mentioned notions. Then we shortly survey results by Rapoport, Richartz [1], by Oort [2], by Mantovan [3], by Arasteh Rad and Hartle [4], by Hartl and Viehmann [5], by Harris and Taylor [6] and by Weiß [7]. More completely we discuss results by Weiß [7]. Let C be a smooth projective geometrically irroducible curve with the function field $\mathbb{F}_q(C)$ over a finite field $\mathbb{F}_q$ with q elements. Let G be a parahoric Bruhat-Tits group scheme over C. Author [7] considers "a foliation structure for Newton strata moduli spaces of bounded global G-shtukas with H-level structure for an arbitrary parahoric Bruhat-Tits group G" and "Igusa varieties". She obtaines a morphism (Main Theorem 0.1) to the moduli space of global G-shtukas. The author then relates here foliation structure to Oort's foliations, to Harris and Taylor and to Mantovan. These results, although difficult to explane in a short reviiew, are well summerised in a short Introduction. Below bounded global G-shtukas with H-level structure are considered. Briefly, the general idea is to start with a foliation stucture on the moduli space of such global G-shtukas and describe it "as a product of a covering of central leaves by Igusa varieties with truncated Rapoport-Zink spaces". The Main Theorem 0.1 gives the morphism from the product of author's Igusa varieties and trancated Rapoport-Zink spaces to the moduli spaces of global G-shtukas. The morphism is finite by the Proposition 6.19. The author also gives an application of the Main Theorem 0.1 to the leaves inside a Newton stratum and compute dimensions of these leaves which turns out to be the same for all leaves. For some details, along with the references above, please see [8, 9].

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Geometry of curves in three-dimensional space and invariants of nonlinear differential equations of the second order

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The talk is devoted to differential invariants of nonlinear differential equations of the second order of the form

$$u_{tt} = [\alpha(u)]_{xx} + [\beta(u)]_x + \gamma(u). \tag{1}$$

where t, x are independent variables and α, β, γ are smooth functions of u.

We considered admissible point transformations only, i.e. transformations of the space of 0-jets $J^0(\mathbb{R}^2)$ that preserve the class of such equations (see, for example, [1, 2]). Admissible transformations form a six-dimensional Lie group with Lie algebra $\mathcal{G}$ thai is generated by the admissible vector fields

$$\frac{\partial}{\partial t}, \quad \frac{\partial}{\partial x}, \quad \frac{\partial}{\partial u} \quad t\frac{\partial}{\partial t}, \quad x\frac{\partial}{\partial x}, \quad u\frac{\partial}{\partial u}. \tag{2}$$

The first three vector fields correspond to translation along the x, y, u axes, and the last three ones correspond to homothety.

Write equation (1) in the following form:

$$u_{tt} = a'(u)u_x^2 - a(u)u_{xx} - b(u)u_x - c(u),$$

where

$$a(u) = \alpha'(u), \quad b(u) = \beta'(u), \quad c(u) = \gamma(u).$$

Consider the following one-dimensional trivial bundle

$$\pi : \mathbb{R}^3 \to \mathbb{R}, \quad \pi : (a, b, c) \mapsto u.$$

A section of this bundle are parametric curve in $\mathbb{R}^3$ that correspond to equation (1). Let $J^k(\pi)$ be the space of k-jets of sections of π with canonical coordinates $u, a_0, b_0, c_0, \ldots, a_k, b_k, c_k$.

Restriction of the Lie algebra of admissible vector fields to the space $J^0(\pi)$ is given by the following vector fields:

$$\frac{\partial}{\partial u}, \quad 2a_0\frac{\partial}{\partial a_0} + b_0\frac{\partial}{\partial b_0}, \quad a_0\frac{\partial}{\partial a_0} + b_0\frac{\partial}{\partial b_0} + c_0\frac{\partial}{\partial c_0}, \quad u\frac{\partial}{\partial u} - c_0\frac{\partial}{\partial c_0}$$

Theorem 1. *The algebra of differential invariants of equations (1) is generated by the following functions:*

$$I_{a,k} = \frac{a_k b_0^{2k}}{a_0^{k+1} c_0^k}, \quad I_{b,k} = \frac{b_k b_0^{2k-1}}{a_0^k c_0^k}, \quad I_{c,k} = \frac{c_k b_0^{2k}}{a_0^k c_0^{k+1}},$$

where $k = 1, 2, \ldots$.

The constructed invariants are analogs of curvature and torsion for curves in three-dimensional Euclidean space.

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On Orthosymmetric n-morphisms

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Let E and F be vector lattices.We say that a bilinear mapping $T : E \times E \to F$ is an orthosymmetric mapping if $T(x,y) = 0$ in F , whenever $\mid x \mid \wedge \mid y \mid = 0$ in E. Generalization of this definition for n linear mapping is that $T : E \times E \times ... \times E \to F$ is an orthosymmetric multilinear mapping if $T(x_1, x_2, ..., x_n) = 0$ for all $x_1, ..., x_n \in E$ such that $\mid x_i \mid \wedge \mid x_j \mid = 0$ for some pair of indices $1 \leq i, j \leq n$. By $E^{\sim}$ we denote the set of all order bounded linear functionals on E. $E_n^{\sim}$ denotes the set of all order continuous linear functionals on E. By $(E^{\sim})_n^{\sim}$ we denote the order continuous order bidual of E. Let $E_1, ..., E_n$ and F be vector lattices. A multilinear mapping $\Psi : E_1 \times ... \times E_n \to F$ is said to be a lattice n-morphism if $\mid \Psi(x_1, ..., x_n) \mid = \Psi(\mid x_1 \mid, ..., \mid x_n \mid)$ for all $x_i \in E_i$ for $i = 1, 2, ..., n$. We say that a lattice n-morphism and orthosymmetric multilinear mapping is an orthosymmetric n-morphism.

Orthosymmetric bilinear mappings have been studied by a lot of authors. For example, M.A. Toumi and R. Yilmaz give the extensions of orthosymmetric bilinear mapping to the order continuous order bidual of a vector lattice by using Arens multiplication.

In this study, we extend an orthosymmetric n-morphism to the order continuous order bidual of a vector lattice by using Arens product. We show that an extension of orthosymmetric n-morphism is again orthosymmetric n-morphism. Unexplained notion and terminology we refer to the following references.

Theorem 1. *Suppose that E is an Archimedean vector lattice and F is a Dedekind complete vector lattice. If $\Psi : E \times E \times ... \times E \to F$ is an orthosymmetric n-morphism, then n-th order adjoint of Ψ on the order continuous order bidual $(E^{\sim})_n^{\sim}$ of E is again an orthosymmetric n-morphism.*

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On homotopy nilpotency of Moore space

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Given based spaces X_1, X_2, we use the customary notations $X_1 \times X_2$ for their Cartesian product, $X_1 \vee X_2$ for their wedge and $X_1 \wedge X_2$ for the smash product of X_1, X_2.

Recall that an *H-space* is a pair (X, μ), where X is a space and $\mu : X \times X \to X$ is a map such that the diagram

$$\begin{tikzcd} X \times X \arrow[r, "\mu"] & X \\ X \vee X \arrow[u, hook] \arrow[ur, "\nabla"'] & \end{tikzcd}$$

commutes up to homotopy, where $\nabla : X \vee X \to X$ is the folding map. An H-space X is called a *group-like space* if X satisfies all the axioms of groups up to homotopy. From now on, we assume that any H-space X is group-like. For an H-space X, we write $\varphi_{X,1} = \iota_X$, $\varphi_{X,2} : X \times X \to X$ for the basic commutator map and $\varphi_{X,n+1} = \varphi_{X,2} \circ (\varphi_{X,n} \times \iota_X)$ for $n \geq 2$.

The *nilpotency class* $\mathrm{nil}\,(X, \mu)$ of an H-space (X, μ) is the least integer $n \geq 0$ for which the map $\varphi_{X,n+1} \simeq *$ is nullhomotopic and we call the homotopy associative H-space X homotopy nilpotent. If no such integer exists, we put $\mathrm{nil}\,(X, \mu) = \infty$. In the sequel, we simply write $\mathrm{nil}\, X$ for the nilpotency class of an H-space X.

In virtue of [2, 2.7. Theorem], we have

Theorem 1. *If X is an H-space then*

$$\mathrm{nil}\, X = \sup_m \mathrm{nil}[X^m, X] = \sup_m \mathrm{nil}[X^{\wedge m}, X] = \sup_Y \mathrm{nil}[Y, X],$$

where m ranges over all integers and Y over all topological spaces.

Then, by means of [8, Lemma 2.6.1], we may state

Corollary 2. *A connected H-space X is homotopy nilpotent if and only if the functor $[-, X]$ on the category of all spaces is nilpotent group valued.*

With any based space X, we associate the integer $\mathrm{nil}\,\Omega(X)$ called the *nilpotency class* of X for the loop space $\Omega(X)$ on X. Although many results on the homotopy nilpotency have been obtained, the homotopy nilpotency classes have been determined in very few cases.

Example 3. (1) It is well-known that

$$\mathrm{nil}\,\Omega(\mathbb{S}^n) = \begin{cases} 3 & \text{for } n \text{ even with } n \neq 2; \\ 2 & \text{for } n \text{ odd with } n \neq 1, 3, 7 \text{ or } n = 2; \\ 1 & \text{for } n = 1, 3, 7 \end{cases}$$

for the n-sphere $\mathbb{S}^n$.

(2) For the wedge $\mathbb{S}^m \vee \mathbb{S}^n$ of two spheres with $m, n \geq 2$, we have

$$\mathrm{nil}\,\Omega(\mathbb{S}^m \vee \mathbb{S}^n) = \infty.$$

Write $\mathbb{K}P^m$ for the projective m-space for $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$, the field of reals or complex numbers and $\mathbb{H}$, the skew $\mathbb{R}$-algebra of quaternions. Then, results from [6] have been applied in [3] to study extensively the homotopy nilpotency of the loop spaces of Grassmann and Stiefel manifolds over $\mathbb{K}$, and their p-localization.

Let $\mathbb{S}^{2m-1}_{(p)}$ be the p-localization of the sphere $\mathbb{S}^{2m-1}$ at a prime p. The main result of the paper [4] is the explicit determination of the homotopy nilpotence class of a wide range of homotopy associative multiplications on localized spheres $\mathbb{S}^{2m-1}_{(p)}$ for $p > 3$.

Next, let A be an Abelian group and n any integer ≥ 2. A CW-complex X satisfying $\pi_j(X) = 0$ for $j < n$, $\pi_n(X) \approx A$ and $H_i(X) = 0$ for $i > n$ is known as a *Moore space* of type (A, n), or simply an $M(A, n)$ space. By [7], it is known that a Moore space $M(A, n)$ with $n \geq 2$ exists and, in view of [5, Example 4.34], the homotopy type of a Moore space $M(A, n)$ is uniquely determined by A and $n \geq 2$. This implies that every Moore space $M(A, n)$ with $n \geq 3$, is the suspension $\Sigma M(A, n-1)$. Furthermore, in [1, Section 2], it was shown that also $M(A, 2)$ is the suspension $\Sigma L(A)$ for some CW-complex $L(A)$.

Now, we examine the homotopy nilpotency of $M(A, n)$ with ≥ 2. Notice that $\mathbb{S}^n = M(\mathbb{Z}, n)$ and the wedge $\mathbb{S}^n \vee \mathbb{S}^n = M(\mathbb{Z} \oplus \mathbb{Z}, n)$ for the integers $\mathbb{Z}$. Then, by Example 3, we have that $\operatorname{nil} \Omega(\mathbb{S}^n) \leq 3$ but $\operatorname{nil} \Omega(\mathbb{S}^n \vee \mathbb{S}^n) = \infty$ for $n \geq 2$.

First, we show the general fact

Proposition 4. *If the reduced homology $\tilde{H}_*(X, \mathbb{F})$ has at least two primitive generators, where $\mathbb{F}$ is a field then $\Omega\Sigma(X)$ is not homotopy nilpotent.*

Then, we state the main resut

Theorem 5. *Let $m \geq 1$, $n_1, \ldots, n_m \geq 2$ and $M(A_k, n_k)$ be Moore spaces of type (A_k, n_k) for $k = 1, \ldots, m$. Then:*

(1) $\operatorname{nil} \Omega((M(A_1, n_1) \times \cdots \times (M(A_m, n_m)) < \infty$ *if and only if if A_k are torsion-free groups with rank* $\mathrm{r}(A_k) = 1$ *for $k = 1, \ldots, m$;*

(2) $\operatorname{nil} \Omega((M(A_1, n_1) \vee \cdots \vee M(A_m, n_m)) < \infty$ *if and only if $m = 1$ and A_1 is a torsion-free group with rank* $\mathrm{r}(A_1) = 1$.

In particular, we derive

Corollary 6. *If $M(A, n)$ is a Moore space with $n \geq 2$ then*

$$\operatorname{nil} \Omega(M(A, n)) < \infty$$

if and only if A is a torsion-free group with rank $r(A) = 1$ or equivalently, A is a subgroup of the rationals $\mathbb{Q}$.

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Metric viewpoint in mapping theory between Riemannian manifolds

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The theory of multidimensional quasiconformal mappings employs three main approaches: analytic, geometric (modulus) and metric ones. In this talk, we use the last approach and establish the relationship between various classes of mappings on Riemannian manifolds including homeomorphisms of finite metric distortion (FMD-homeomorphisms), finitely bi-Lipschitz, quasisymmetric and quasiconformal mappings. The appropriate classes of homeomorphisms involving the modulus technique are also presented. One of the main results shows that FMD-homeomorphisms are lower Q-homeomorphisms. As an application, there are obtained some sufficient conditions for boundary extensions of FMD-homeomorphisms. These conditions are illustrated by several examples of FMD-homeomorphisms.

A classical example of significance of metric approach can be illustrated by the Bohr-Menchoff-Trokhymchuk theory on analyticity (monogeneity) of a complex variable function. In 1937 Menchoff [2] generalized the Bohr theorem [1] on analytic functions in the terms of preserving infinitesimal circles. More precisely, for a continuous and locally univalent mapping $w = f(z)$ of a domain D onto a domain $D*$ and $z_0 \in D$, take the quantity

$$H(z_0, r) = \frac{\max\limits_{|z'-z_0|=r} |f(z') - f(z_0)|}{\min\limits_{|z''-z_0|=r} |f(z'') - f(z_0)|}$$

and say that f preserves infinitesimal circles in D if $H(z_0, r) \to 1$ as $r \to 0$. The Menchoff result states that the preserving infinitesimal circles at all z_0 except for at most a countable set completely provides that either f or its conjugate is analytic in D. This pure metric condition has been extended to continuous mappings by Yu. Yu. Trokhymchuk [3] involving the Stoilow theory on interior mappings.

The classes of mappings presented in the talk can be treated as far advanced extensions of the Bohr-Menchoff-Trokhymchuk theory on complex plane to more general structures.

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Monogenic functions with values in commutative complex algebras of the second rank with unity and generalized biharmonic equation with non-zero simple characteristics

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Among all two-dimensional algebras of the second rank with unity e over the field of complex numbers $\mathbb{C}$, we found a semi-simple algebra $\mathbb{B}_0 := \{c_1 e + c_2 \omega : c_k \in \mathbb{C}, k = 1, 2\}$, $\omega^2 = e$, containing bases $\{e_1, e_2\}$, such that $\mathbb{B}_0$-valued "analytic" functions $\Phi(xe_1 + ye_2)$ (x, y are real variables) satisfy the fourth order homogeneous partial differential equation of the form:

$$\left(b_1 \frac{\partial^4}{\partial y^4} + b_2 \frac{\partial^4}{\partial x \partial y^3} + b_3 \frac{\partial^4}{\partial x^2 \partial y^2} + b_4 \frac{\partial^4}{\partial x^3 \partial y} + b_5 \frac{\partial^4}{\partial x^4}\right) u(x, y) = 0, \tag{1}$$

where complex coefficients $b_k \in \mathbb{C}$, $k = \overline{1, 5}$, $b_5 \neq 0$, such than the Eq. of characteristics

$$l(s) := b_1 s^4 + b_2 s^3 + b_3 s^2 + b_4 s + b_5 = 0, s \in \mathbb{C}, \tag{2}$$

has four pairwise different roots (each root is a simple root).

A set of pairs $(\{e_1, e_2\}, \Phi)$, where all real components of Φ satisfy Eq. 1, is described in the explicit form.

A totalies of "analytic" functions $\Phi(xe_1 + ye_2)$, such that the first real component of each of them satisfies the given solution u of Eq. 1 in the simply-connected bounded domains, are found in [2, 3, 5, 6].

Particular cases of this research are considered in [1, 2, 3, 4, 5].

The complete statements, proofs and definitions are considered in [6].

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The Collatz conjecture from an algebraic point of view

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The Collatz conjecture is an open problem in number theory stablished in 1937 by Lothar Collatz and can be stated as follows: If $f : \mathbb{N} \to \mathbb{N}$ is the function defined by:

$$f(n) = \begin{cases} \frac{n}{2} & ; n \textit{ is even} \\ 3n+1 & ; n \textit{ is odd} \end{cases}$$

the conjecture says that given $n \in \mathbb{N}$, there exists $k > 0$ such that $f^{(k)}(n) = 1$ and the only orbit is $\{1, 2, 4\}$

Every topology τ can be seen as a commutative semiring under union and intersection. If τ_f is the topology on $\mathbb{N}$ given by the open sets as those subset θ of $\mathbb{N}$ such that $f^{-1}(\theta) \subset \theta$, we prove that the Collatz conjecture is true if and only if τ_f, viewed as a commutative semiring, is a local semiring.

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On the monoid of cofinite partial isometries of positive integers with a bounded finite noise

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We follow the terminology of [2, 4, 5]. For any positive integer j the semigroup $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ is called the *monoid of cofinite isometries of positive integers with the noise* j. It was introduced in [4].

Any inverse semigroup S admits the *minimum group congruence* $\mathfrak{C}_{\mathbf{mg}}$:

$$a\mathfrak{C}_{\mathbf{mg}}b \quad \text{if and only if} \quad \text{there exists} \quad e \in E(S) \quad \text{such that} \quad ea = eb.$$

Proposition 1. *Let γ and δ be elements of the monoid $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$. Then $\gamma\mathfrak{C}_{\mathbf{mg}}\delta$ in $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ if and only if $n_{\gamma}^{\mathbf{r}} - n_{\gamma}^{\mathbf{d}} = n_{\delta}^{\mathbf{r}} - n_{\delta}^{\mathbf{d}}$. Moreover, the quotient semigroup $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}/\mathfrak{C}_{\mathbf{mg}}$ is isomorphic to the additive group of integers $\mathbb{Z}(+)$ by the map*

$$\pi_{\mathfrak{C}_{\mathbf{mg}}} \colon \mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]} \to \mathbb{Z}(+), \quad \gamma \mapsto n_{\delta}^{\mathbf{r}} - n_{\delta}^{\mathbf{d}}.$$

Example 2. We put $\mathcal{C}\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]} = \mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]} \sqcup \mathbb{Z}(+)$ and extend the multiplications from $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ and $\mathbb{Z}(+)$ onto $\mathcal{C}\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ in the following way:

$$k \cdot \gamma = \gamma \cdot k = k + (\gamma)\pi_{\mathfrak{C}_{\mathbf{mg}}} \in \mathbb{Z}(+), \qquad \text{for all} \quad k \in \mathbb{Z}(+) \quad \text{and} \quad \gamma \in \mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}.$$

Theorem 3. *For any positive integer j every Hausdorff shift-continuous topology τ on $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ is discrete.*

Proposition 4. *Let j be any positive integer and $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ be a proper dense subsemigroup of a Hausdorff semitopological semigroup S. Then $I = S \setminus \mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ is a closed ideal of S.*

Theorem 5. *Let j be any positive integer and $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ be a proper dense subsemigroup of a Hausdorff topological inverse semigroup S. Then $I = S \setminus \mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ is a topological group.*

Corollary 6. *Let j be any positive integer and $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ be a proper dense subsemigroup of a Hausdorff topological inverse semigroup S. Then the group $S \setminus \mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ contains a dense cyclic subgroup.*

Example 7. Let $\mathcal{C}\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ be a semigroup defined in Example 2. Put M be an arbitrary subset of $\{2, \ldots, j\}$. We define the topology τ_{lc}^{M} on $\mathcal{C}\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ in the following way:

(i) all elements of the monoid $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ are isolated points in $\left(\mathcal{C}\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}, \tau_{\mathsf{lc}}^{M}\right)$;

(ii) for any $k \in \mathbb{Z}(+)$ the family $\mathscr{B}_{\mathsf{lc}}^{M}(k) = \left\{U_i^M(k) \colon i \in \mathbb{N}\right\}$, where

$$U_i^M(k) = \{k\} \cup \left\{\gamma \in \mathcal{C}\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}[M] \colon k \preccurlyeq \gamma \text{ and } n_{\gamma}^{\mathbf{d}} \geqslant i\right\},$$

is the base of the topology τ_{lc}^{M} at the point $k \in \mathbb{Z}(+)$.

Theorem 8. *Let j be any positive integer and $\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$ be a proper dense subsemigroup of a Hausdorff locally compact topological inverse semigroup (S, τ). Then (S, τ) topologically isomorphic to the topological inverse semigroup $\left(\mathcal{C}\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}, \tau_{\mathsf{lc}}^{M}\right)$ for some subset M of $\{2, \ldots, j\}$.*

Corollary 9. *For any positive integer j there exists $(j-1)!+1$ distinct topologically non-isomorphic Hausdorff locally compact semigroup inverse topologies on the monoid $\mathcal{C}\mathbf{I}\mathbb{N}_{\infty}^{\boldsymbol{g}[j]}$.*

The obtained results generalize the corresponding results of the papers [1] and [3].

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On some generalization of the bicyclic monoid

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We introduce algebraic extensions $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ of the bicyclic monoid for an arbitrary ω-closed family $\mathscr{F}$ subsets of ω which generalizes the bicyclic monoid, the countable semigroup of matrix units and some other combinatorial inverse semigroups. It is proven that $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is combinatorial inverse semigroup and Green's relations, the natural partial order on $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ and its set of idempotents are described. We prove the criteria of simplicity, 0-simplicity, bisimplicity, 0-bisimplicity of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$. We gave the criteria when the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ has the identity, and when the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is isomorphic to the bicyclic semigroup or the countable semigroup of matrix units.

Variational principles for metric mean dimension

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Entropy has been a fundamental concept in the theory of dynamical systems from its beginnings. Together with the newer concept of mean dimension, these invariants can be related to various embeddings in shift spaces. An important result from 1970, known as the "variational principle" relates topological and measurable entropies. Recently various variational principles relating metric mean dimension and (variants of) measurable entropy have been proven. We will survey some of these old and new developments. Based on joint work with Adam Śpiewak.

A generalized Lie-algebraic approach to constructing of integrable fractional dynamical systems

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In the paper [1] a new metrized Lie algebra of fractional integral-differential operators has been introduced and the infinite hierarchy of Lax type Hamiltonian flows on its dual space, which is reduced to the hierarchies of the Lax integrable fractional-differential dynamical systems on coadjoint orbits, has been constructed by use of the Adler-Kostant-Symes Lie-algebraic scheme. In our report we propose a generalization of the described in [1] Lie-algebraic approach to constructing of Lax integrable fractional-differential dynamical systems, which is based on the central extension by the Maurer-Cartan 2-cocycle of the mentioned above operator Lie algebra. By means of this generalized approach we obtain the Lax integrable fractional-differential Kadomtsev-Petviashvily hierarchy, whose quasi-classical approximation leads to the Benney type hydrodynamic systems.

Let us consider the Lie algebra $\mathbb{A}_\alpha := \mathbb{A}_0\{\{D^\alpha, D^{-\alpha}\}\}$ (see [1]), which consists of the fractional integral-differential operators in the forms:

$$\mathrm{a}_\alpha := \sum\nolimits_{j\in\mathbb{Z}_+} a_j D^{\alpha(m_\alpha - j)},$$

where $\mathbb{A}_0 := A\{\{D, D^{-1}\}\}$ is the Lie algebra of integral-differential operators, $A := W_2^\infty(\mathbb{R};\mathbb{C})\cap W_\infty^\infty(\mathbb{R};\mathbb{C})$, $D^\alpha : A \to A$ is a Riemann-Liouville fractional derivative, $\alpha \in \mathbb{C}\setminus\mathbb{Z}$, $\operatorname{Re}\alpha \neq 0$, $m_\alpha \in \mathbb{Z}_+$ and $a_j \in \mathbb{A}_0$, $j \in \mathbb{Z}_+$, and possesses the standard commutator $[.,.]$ and invariant with respect to this commutator scalar product:

$$(\mathrm{a}_\alpha, \mathrm{b}_\alpha) := \int_{\mathbb{R}} \operatorname{res}_D\left(\operatorname{res}_{D_\alpha}\left(\mathrm{a}_\alpha \circ \mathrm{b}_\alpha D^{-\alpha}\right)\right)dx,$$

where $\operatorname{res}_{D_\alpha}$ denotes a coefficient at $D^{-\alpha}$ for any fractional integral-differential operator as well as res_D denotes a coefficient at D^{-1} for any integral-differential operator. The Lie algebra $\mathbb{A}_\alpha$ allows the splitting into the direct sum of its two Lie subalgebras $\mathbb{A}_\alpha = \mathbb{A}_{\alpha,+} \oplus \mathbb{A}_{\alpha,-}$, where $\mathbb{A}_{\alpha,+}$ is the Lie subalgebra of the formal power series by the operator D^α.

One parameterizes the Lie algebra $\mathbb{A}_0$ by the variable $y \in \mathbb{S}^1$ and constructs the central extension $\hat{\mathbb{A}}_\alpha := \bar{\mathbb{A}}_\alpha \oplus \mathbb{C}$ of the Lie algebra $\bar{\mathbb{A}}_\alpha := \prod_{y\in\mathbb{S}^1}\mathbb{A}_\alpha$ by the Maurer-Cartan 2-cocycle $\omega_2(.,.)$ on $\bar{\mathbb{A}}_\alpha$ with the commutator:

$$[(\mathrm{a}_\alpha, d), (\mathrm{b}_\alpha, e)] = ([\mathrm{a}_\alpha, \mathrm{b}_\alpha], \omega_2(\mathrm{a}_\alpha, \mathrm{b}_\alpha)), \qquad (\mathrm{a}_\alpha, d), (\mathrm{b}_\alpha, e) \in \hat{\mathcal{L}}_\alpha, \tag{1}$$

$$[\mathrm{a}_\alpha, \mathrm{b}_\alpha] = \mathrm{a}_\alpha \circ \mathrm{b}_\alpha - \mathrm{b}_\alpha \circ \mathrm{a}_\alpha, \qquad \omega_2(\mathrm{a}_\alpha, \mathrm{b}_\alpha) := \int_{\mathbb{S}^1} (\mathrm{a}_\alpha, \partial \mathrm{b}_\alpha/\partial y)dy.$$

The invariant with respect to the commutator (1) scalar product on $\hat{\mathbb{A}}_\alpha$ is given by the relationship:

$$((\mathrm{a}_\alpha, d), (\mathrm{b}_\alpha, e)) = \int_{\mathbb{S}^1} (\mathrm{a}_\alpha, \mathrm{b}_\alpha)dy + ed.$$

The Lie-Poisson bracket, deformed by the space endomorphism $\mathcal{R} = (P_+ - P_-)/2 : \bar{\mathbb{A}}_\alpha \to \bar{\mathbb{A}}_\alpha$, takes the form:

$$\{\gamma, \mu\}_{\mathcal{R}}(\tilde{l}_\alpha) = (\tilde{l}_\alpha, [\mathcal{R}\nabla\gamma(\tilde{l}_\alpha), \nabla\mu(\tilde{l}_\alpha)] + [\nabla\gamma(\tilde{l}_\alpha), \mathcal{R}\nabla\mu(\tilde{l}_\alpha)])+$$

$$+c\omega_2(\mathcal{R}\nabla\gamma(\tilde{l}_\alpha),\nabla\mu(\tilde{l}_\alpha))+c\omega_2(\nabla\gamma(\tilde{l}_\alpha),\mathcal{R}\nabla\mu(\tilde{l}_\alpha)),$$

where $\gamma,\mu\in\mathcal{D}(\bar{\mathbb{A}}_\alpha^*)$ are smooth by Frechet functionals on $\bar{\mathbb{A}}_\alpha^*\simeq\bar{\mathbb{A}}_\alpha$, $\tilde{l}_\alpha\in\bar{\mathbb{A}}_\alpha^*$, $c\in\mathbb{C}$, $P_\pm$ being projectors on $\mathbb{A}_{\alpha,\pm}$, and generate the infinite hierarchy of Lax type Hamiltonian flows:

$$\partial\tilde{l}_\alpha/\partial t_j=[(\nabla\gamma_j(\tilde{l}_\alpha))_+,\tilde{l}_\alpha-c\partial/\partial y],\quad(\tilde{l}_\alpha,c)\in\hat{\mathcal{L}_\alpha}^*,\quad j\in\mathbb{N},\tag{2}$$

by means of the Casimir invariants $\gamma_j\in I(\bar{\mathbb{A}}_\alpha^*)$, $j\in\mathbb{N}$, as Hamiltonians. The Casimir invariants satisfy the relationship:

$$[\tilde{l}_\alpha-c\partial/\partial y,\nabla\gamma_j(\tilde{l}_\alpha)]=0,$$

and can be found in the forms:

$$\gamma_j(\tilde{l}_\alpha)=\int_{y\in\mathbb{S}^1}Tr(\tilde{l}_\alpha^0D^{j\alpha})dy,\quad j\in\mathbb{N}.$$

Here the coefficients of the operator $\tilde{l}_\alpha^0:=D^{j\alpha}+\sum_{k\le j-1}\hat{C}_kD^{k\alpha}$, $k\in\mathbb{Z}$, are such that $d\hat{C}_k/dx=0=d\hat{C}_k/dt_j$, $j\in\mathbb{N}$, and obey the equality:

$$(\tilde{l}_\alpha-c\partial/\partial y)\circ\Phi=\Phi\circ(\tilde{l}_\alpha^0-c\partial/\partial y),\quad\Phi=1+\sum\nolimits_{r\in\mathbb{N}}\Phi_rD^{-r\alpha}.$$

As an example, one studies the reduction of the hierarchy (2) on the coadjoint orbit related with the element

$$\tilde{l}_\alpha=D^{2\alpha}+D^\alpha\hat{v}+\hat{v}D^\alpha+\hat{u}\in\bar{\mathbb{A}}_\alpha^*,$$

where $\hat{u},\hat{v}\in\bar{\mathbb{A}}_0$, when $c=1$. Looking for the gradients of the Casimir invariants in the forms $\nabla\gamma_j(\tilde{l}_\alpha)=D^{m\alpha}+\sum_{k\le j-1}\hat{a}_{k,j}D^{k\alpha}$, $m\in\mathbb{Z}_+$, $k\in\mathbb{Z}$, one obtains the hierarchy of fractional-differential dynamical systems such as

$$\begin{aligned}&d\hat{u}/dt_1=\hat{u}_y+[\hat{v},\hat{u}], && d\hat{u}/dt_2=\hat{u}_y,\\ &d(\hat{v}+D^\alpha\hat{v}D^{-\alpha})/dt_1=[D^\alpha,\hat{u}-\hat{v}^2]D^{-\alpha}, && d(\hat{v}+D^\alpha\hat{v}D^{-\alpha})/dt_2=\hat{v}_y,\end{aligned}$$

$$\begin{aligned}&d\hat{u}/dt_3=\hat{f}_y+[\hat{f},\hat{u}]\\ &d(\hat{v}+D^\alpha\hat{v}D^{-\alpha})/dt_3=\hat{q}_y+([\hat{q}D^\alpha,\hat{u}]+[D^\alpha\hat{u}+\hat{v}D^\alpha\hat{v},\hat{u}]+[\hat{f},\hat{v}D^\alpha+D^\alpha\hat{v}])D^{-\alpha},\\ &[\hat{q},D^{2\alpha}]=D^\alpha[D^\alpha,\hat{u}-\hat{v}^2],\\ &[\hat{f},D^{2\alpha}]=-(\hat{v}_yD^{2\alpha}+D^\alpha\hat{v}_yD^\alpha+D^{2\alpha}\hat{v}_y)-[\hat{v}D^{2\alpha}+D^{2\alpha}\hat{v},\hat{u}]-[D^\alpha\hat{u},\hat{v}D^\alpha+D^\alpha\hat{v}]-\\ &\qquad-[\hat{u}D^\alpha,D^\alpha\hat{v}]-[\hat{v},D^\alpha\hat{v}^2D^\alpha]-[\hat{q}D^\alpha,\hat{v}D^\alpha+D^\alpha\hat{v}],\\ &\qquad\qquad .....\,.\end{aligned}\tag{3}$$

The gradients of corresponding Casimir invariants are written as

$$\begin{aligned}&\nabla\gamma_1(\tilde{l}_\alpha)=D^\alpha+\hat{v}+\sum\nolimits_{k\le0}\hat{a}_{k,1}D^{k\alpha},\\ &\nabla\gamma_2(\tilde{l}_\alpha)=D^{2\alpha}+(\hat{v}+D^\alpha\hat{v}D^{-\alpha})D^\alpha+\hat{u}+\sum\nolimits_{k\le0}\hat{a}_{k,2}D^{k\alpha},\\ &\nabla\gamma_3(\tilde{l}_\alpha)=D^{3\alpha}+(\hat{v}+D^\alpha\hat{v}D^{-\alpha}+D^{2\alpha}\hat{v}D^{-2\alpha})D^{2\alpha}+\hat{b}D^\alpha+\hat{f}+\sum\nolimits_{k\le0}\hat{a}_{k,3}D^{k\alpha},\quad .....\,,\end{aligned}$$

where $\hat{b}=\hat{u}+D^\alpha\hat{u}D^{-\alpha}+\hat{v}D^\alpha\hat{v}D^{-\alpha}+\hat{q}$. The third system in the hierarchy (3) can be considered as a fractional-differential analog of the Kadomtsev-Petviashvily equation.

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On the solution of separate differential equations with variational derivatives of the first and second orders

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The theory of equations with variational (functional) derivatives is a fairly extensive area of mathematics. This class of equations has numerous applications in statistical physics, quantum field theory, hydromechanics and other fields. The theory of variational derivatives, differential and integro-differential equations with variational derivatives is quite fully stated, for example, in the monographs [1–3] and the works [4–10].

The problems that are formulated and investigated in this area of mathematics are similar to the problems considered in the case of ordinary and partial differential equations. Explicit formulas for solving equations with variational derivatives are known only in a few cases. This applies mainly to the set of linear equations [5–8]. Therefore, the main methods for solving such equations are approximate.

The problem of an approximate solution of equations with variational derivatives is not sufficiently studied. When solving this class of problems, it may be useful to apply methods that take into account the given initial and boundary values. In particular, in the Cauchy problem for an equation of the n-th order, the desired functional $F(x)$ can be approximately found from the values of the functional $F(x_0)$ and its variational derivatives up to the $(n-1)$-th order, which are known at the point $x_0(t)$. For this, it is natural to use the operator interpolation apparatus [11, 12]. Consider one of the ways for the approximate solution of equations with variational derivatives, based on interpolation of the functional included in the equation.

We formulate the definition of the variational derivative for functionals defined on sets of functions [9]. Let X be a linear space of real functions defined on a segment $[a, b]$ of the real axis $\mathbb{R}$, and F be an operator or functional given on the X.

The k-th order Gateaux differential $\delta^k F[x; h_1, h_2, ..., h_k]$ $(k \in \mathbb{N})$ of the mapping F at the point $x \in X$ in the directions $h_1, h_2, ..., h_k \in X$ is defined by the equality

$$\delta^k F[x; h_1, h_2, ..., h_k] = \lim_{\lambda \to 0} \frac{\delta^{k-1} F[x + \lambda h_k; h_1, h_2, ..., h_{k-1}] - \delta^{k-1} F[x; h_1, h_2, ..., h_{k-1}]}{\lambda} =$$

$$= \left. \frac{\partial^k F\left(x + \lambda_1 h_1 + \lambda_2 h_2 + ... + \lambda_k h_k\right)}{\partial \lambda_k \cdots \partial \lambda_1} \right|_{\lambda_1 = ... = \lambda_k = 0}, \delta^0 F[x] \equiv F\left(x\right).$$

If there exists the k-th order Gateaux differential $\delta^k F[x; h_1, h_2, ..., h_k]$ $(x, h_i \in X;\ i = 1, 2, ..., k)$ of the functional $F\left(x\right)$ at the point $x \in X$ in the directions $h_1, h_2, ..., h_k \in X$, that can be represented as

$$\delta^k F[x; h_1, h_2, ..., h_k] = \int_{[a,b]^k} a\left(x; t_1, ..., t_k\right) h_1\left(t_1\right) ... h_k\left(t_k\right) dt_1 ... dt_k, \quad (1)$$

where $a\left(x; t_1, ..., t_k\right)$ is some function depending on $x = x\left(s\right)$ and variables $t_1, ..., t_k \in \mathbb{R}$, then $a\left(x; t_1, ..., t_k\right)$ is called the variational derivative of the k-th order of the functional $F\left(x\right)$ with respect to x at the point $t = (t_1, t_2, ..., t_k)$ and denoted by the symbol $\dfrac{\delta^k F(x)}{\delta x(t_1) \cdots \delta x(t_k)}$. Variational derivatives can be generalized functions and other types of functionals.

As X, one can choose the space $C[a,b]$ of continuous functions with a uniform norm, the Hilbert space $L_2[a,b]$ or any other space such that the integral on the right-hand side of (1) makes sense.

We give formulas for the exact solution of some of the simplest differential equations with variational derivatives.

For example, for the equation

$$\frac{\delta F(x)}{\delta x(t)} = 2p\,(t)\cos x\,(t)\int_0^1 p\,(t)\sin x\,(t)\,dt,$$

the solution is the functional $F\,(x) = \left(\int_0^1 p\,(t)\sin x\,(t)\,dt\right)^2$, where $p\,(t)$ and $x\,(t)$ are elements of space $C[0,1]$.

The functional $F\,(x) = \int_a^b p\,(t)\,f\,[x\,(t)]\,dt$ is the solution of the equation $\frac{\delta F(x)}{\delta x(t)} = p\,(t)\,f'\,[x\,(t)]$.

The solution of the equation

$$\frac{\delta F(x)}{\delta x(t)} = p_0\,(t)\,e^{x(t)} + p_1(t)\cos x\,(t) + p_2\,(t)\sin x\,(t) + p_3\,(t)\,x\,(t) + p_4\,(t)\int_0^1 p_4\,(\tau)\,x\,(\tau)\,d\tau$$

has the form

$$F\,(x) = \int_0^1 \left[p_0\,(t)\,e^{x(t)} + p_1\,(t)\sin x\,(t) - p_2(t)\cos x\,(t) + \frac{1}{2}\left(\int_0^1 p_4\,(t)\,x\,(t)\,dt\right)^2\right] p\,(t)\sin x\,(t)\,dt,$$

where $p_0\,(t)$ $(i = 0, 1, ..., 4)$ are arbitrary functions for which reduced integrals exist.

Next, we consider a differential equation of the hyperbolic type with the second-order variational derivatives:

$$\frac{\delta^2 u\,(x,y)}{\delta x^2\,(t)} - a^2\,(t)\,\frac{\delta^2 u\,(x,y)}{\delta y^2\,(t)} = 0\,(x = x\,(t) \geq 0,\ y = y\,(t)\,,\ a\,(t) \neq 0;\ t \in [a,b] \subseteq \mathbb{R})\,. \qquad (2)$$

The solution of this equation is the functional

$$u\,(x,y) = f_1\left[\int_a^b (y\,(t) + a\,(t)\,x\,(t))\,dt\right] + f_2\left[\int_a^b (y\,(t) - a\,(t)\,x\,(t))\,dt\right], \qquad (3)$$

where $f_1\,(\cdot)$ è $f_2\,(\cdot)$ are any functions that are twice differentiable on $\mathbb{R}$. The representation (3) is an analogue of the classical Dalamber formula.

We give the Hermite interpolation formula $H\,(x,y)$ with respect to a single node of the second multiplicity, which is an approximation to the solution $u\,(x,y)$ of the Cauchy problem for equation (2) with the initial conditions

$$u\,(x_0,y) = u_0\,(y)\,,\ \frac{\delta u\,(x_0,y)}{\delta x(t)} = u_1\,(y)\,, \qquad (4)$$

where $u_0\,(y)$ and $u_1\,(y)$ are some functionals defined on $C[a,b]$.

Theorem 1. *An approximate solution of the Cauchy problem (2), (4) can be represented as*

$$H\,(x,y) = u_0\,(y)\ + u_1\,(y)\int_a^b\ (x(t) - x_0(t))\,dt + \frac{1}{2}a^2\,(t)\,u_0''\,(y)\,[x(t) - x_0(t)]^2\,. \qquad (5)$$

The proof of this theorem is based on a direct verification of the interpolation conditions (4).

Substituting the approximation $H\,(x,y)$ of the form (5) to the solution $u\,(x,y)$ of the equation (2) in the left-hand side of equality (2), we obtain

$$\frac{\delta^2 H\,(x,y)}{\delta x^2\,(t)} - a^2\,(t)\,\frac{\delta^2 H\,(x,y)}{\delta y^2\,(t)} =$$

$$= -a^2(t)\left(u_1''(y)\int_a^b (x(t)-x_0(t))\,dt + \frac{1}{2}a^2(t)\,u_0^{(4)}(y)\,[x(t)-x_0(t)]^2\right)\delta^2(t-s),$$

where the delta function $\delta(t) = \begin{cases} 0, & t \neq 0; \\ +\infty, & t = 0. \end{cases}$ We note that in the case $t \neq s$, the value $\delta(t-s) = 0$ and the equality (2) takes place for any (x, y) from the domain of definition.

The obtained results can serve as a basis for further research of the theory of differential equations with variational derivatives that is not well developed, and can also be used to construct approximate interpolation methods for solving some linear and nonlinear differential equations with variational derivatives of the first and second order that are found in various applied fields and mathematical physics.

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Special mean and total curvature of a dual surface in isotropic spaces

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In this paper, we study the properties of the total and mean curvatures of a surface and its dual image in an isotropic space. We prove the equality of the mean curvature and the second quadratic forms. The relation of the mean curvature of a surface to its dual surface is found. The superimposed space method is used to investigate the geometric characteristics of a surface relative to the normal and special normal.

Consider an affine space A_3 with the coordinate system $Oxyz$. Let $\vec{X}(x_1, y_1, z_1)$ and $\vec{Y}(x_2, y_2, z_2)$ be vectors of A_3.

Definition 1. If the scalar product of the vectors $\vec{X}$and $\vec{Y}$ is defined by the formula

$$\begin{cases} (X,Y)_1 = x_1x_2 + y_1y_2 & \text{if} \quad x_1x_2 + y_1y_2 \neq 0, \\ (X,Y)_2 = z_1z_2 & \text{if} \quad x_1x_2 + y_1y_2 = 0, \end{cases} \tag{1}$$

then A_3 is said to be an isotropic space R_3^2. [1, 2]

Geometry in a plane of an isotropic space will be Euclidean if it is not parallel to the oz axis. When a plane is parallel to oz, the geometry on it will be Galilean.

Since an isotropic space has an affine structure, there is an affine transformation that preserves the scalar product by formula (1). This motion of an isotropic space is given by the formula [5]

$$\begin{cases} x' = x\cos\alpha - y\sin\alpha + a \\ y' = x\sin\alpha + y\cos\alpha + b \\ z' = Ax + By + z + c \end{cases} \tag{2}$$

The second sphere is defined as a surface with the constant normal curvature. This sphere of the unit radius has the equation [8]

$$x^2 + y^2 = 2z, \tag{3}$$

we call it the isotropic sphere.

Let a plane π be given in R_3^2, which is not parallel to the oz axis of the space. Consider the section of the isotropic sphere by the plane π and denote it by Γ. Since an isotropic sphere is a paraboloid of revolution, the section Γ by a plane is a closed curve. It was proved in [2] that Γ is an ellipse.

Draw tangent planes to isotropic sphere (3) through points$M \in \Gamma$. Denote the set of tangent planes to points F by $\{\pi\}$.

The following statement holds.

Theorem 2. *All planes of the set $\{\pi\}$ intersect at one point.* [6]

If a plane π_0 is given by the equation

$$z = Ax + By + C, \tag{4}$$

then the intersection point of the planes of the set $\{\pi\}$ will be $(A, B, -C)$.

Definition 3. The point $(A, B, -C)$ will be called dual to plane (4) with respect to isotropic sphere (3). [6]

Let us draw the tangent plane π_M to the surface F at the point $M(x_0, y_0, z_0)$. Denote by M^* the dual image of the tangent space π_M with respect to the isotropic sphere. When the point $M \in F$ changes on the surface F, its dual image describes a surface F^*.

Definition 4. . The surface F^* is said to be the dual surface to the surface F in an isotropic space. [6]

When F is given by the equation $z = f(x, y)$, F^* has the equations

$$\begin{cases} x^*(u,v) = f'_u(u,v) \\ y^*(u,v) = f'_v(u,v) \\ z^*(u,v) = u \cdot f'_u(u,v) + v \cdot f'_u(u,v) - f(u,v) \end{cases} \tag{5}$$

Lemma 5. *When the total curvature of a surface $K = 0$, its dual image is a point or a curve.*

Theorem 6. *The product of the total curvatures of the surface $F$and the dual surface F^* of the isotropic space is equal to unity:*

$$K \cdot K^* = 1. \tag{6}$$

Lemma 7. *The special mean curvatures of the surfaces, given by the functions $\overrightarrow{R_1}(u,v) = f_u \cdot \overrightarrow{i} + f_v \cdot \overrightarrow{j} + f_u \cdot \overrightarrow{k}$ and $\overrightarrow{R_2}(u,v) = f_u \cdot \overrightarrow{i} + f_v \cdot \overrightarrow{j} + f_v \cdot \overrightarrow{k}$, are calculated, respectively, by the formulas*

$$H_m(R_1) = \frac{f_{uvv}\left(f_{uu}^2 + f_{uv}^2\right) - 2f_{uuv}\left(f_{uu}f_{uv} + f_{uv}f_{vv}\right) + f_{uuu}\left(f_{uv}^2 + f_{vv}^2\right)}{\left[f''_{uu}f''_{vv} - {f''_{uv}}^2\right]^2}, \tag{7}$$

$$H_m(R_2) = \frac{f_{vvv}\left(f_{uu}^2 + f_{uv}^2\right) - 2f_{uvv}\left(f_{uu}f_{uv} + f_{uv}f_{vv}\right) + f_{uuv}\left(f_{uv}^2 + f_{vv}^2\right)}{\left[f''_{uu}f''_{vv} - {f''_{uv}}^2\right]^2}. \tag{8}$$

Lemma 8. *The mean curvatures of the surfaces, given by the functions $\overrightarrow{R_1}(u,v)$ and $\overrightarrow{R_2}(u,v)$, are equal to zero.*

Lemma 9. *The mean curvature and special mean curvature of the dual surface* (5) *and the surfaces $R_1(u,v)$, $R_2(u,v)$ are connected by the equality:*

$$H_m^* = H^* + u \cdot H_m(R_1) + v \cdot H_m(R_2). \tag{9}$$

Theorem 10. *The mean curvatures defined with respect to the normal and the special normal are equal: $H_m^* = H^*$.*

Theorem 11. *If $\Omega = 0$, then the special total curvature of the surface F^* is expressed in terms of the special total curvatures of the surfaces F, Z_1, and Z_2.*

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A topological transformation group of a hyperspace

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Let X be a compact Hausdorff space. By $\exp X$ we denote a set of all nonempty closed subsets of X. A family of sets of the view

$$O\langle U_1, ..., U_n\rangle = \{F \in \exp X : \ F \subset \bigcup_{i=1}^{n} U_n, F \cap U_1 \neq \varnothing, ..., F \cap U_n \neq \varnothing\}$$

forms a base of a topology on $\exp X$, where $U_1, \ldots, U_n$ are open nonempty sets in X. This topology is called *the Vietoris topology*. A space $\exp X$ equipped with Vietoris topology is called *hyperspace* of X. For a compact space X its hyperspace $\exp X$ is also a compact space.

Let $f\colon X \to Y$ be continuous map of compacts, $F \in \exp X$. We put

$$(\exp f)(F) = f(F).$$

This equality defines a map $\exp f\colon \exp X \to \exp Y$. For a continuous map f the map $\exp f$ is continuous. Really, it follows from the formula [2]

$$(\exp f)^{-1} O\langle U_1, ..., U_m\rangle = O\langle f^{-1}(U_1), ..., f^{-1}(U_m)\rangle$$

what one can check directly. Note that if $f\colon X \to Y$ is an epimorphism, then $\exp f$ is also an epimorphism.

For a Tychonoff space X we put

$$\exp_\beta X = \{F \in \exp \beta X : \ F \subset X\}.$$

It is clear, that $\exp_\beta X \subset \exp \beta X$. Consider the set $\exp_\beta X$ as a subspace of the space $\exp \beta X$. For a Tychonoff spaces X the space $\exp_\beta X$ is also a Tychonoff space with respect to the induced topology.

For a continuous map $f\colon X \to Y$ of Tychonoff spaces we put

$$\exp_\beta f = (\exp \beta f)|_{\exp_\beta X},$$

where $\beta f\colon \beta X \to \beta Y$ is the Stone-Cěch compactification of f (it is unique).

For a Tychonoff space X put

$$\exp(\mathrm{Homeo}\,(X)) = \{\exp(g) : \ g \in \mathrm{Homeo}\,(X)\}.$$

Proposition 1. *For an arbitrary Tychonoff space X we have*

$$\exp(\mathrm{Homeo}\,(X)) \subset \mathrm{Homeo}\,(\exp(X)).$$

Note that the inclusion cannot be reversed.

Example 1. Let $X = \{a, b\}$ be a two-point discrete space. Then $\exp X$ is three point discrete space. There exist only two homeomorphisms of X onto itself: h, $h'\colon X \to X$, defined by the rules $h(a) = a$, $h(b) = b$ and $h'(a) = b$, $h'(b) = a$. At the same time $\exp X$ has six different homeomorphisms four of them could not be generated by h and h'.

For a topological transformation group (G, X, α) we put

$$\exp(G) = \{\exp(\alpha_g) : g \in G\},$$

here $\alpha_g(x) = g(x)$.

Let U_g be an open in G neighbourhood of an element $g \in G$. we define a set $U_{\exp(\alpha_g)} = \{\exp(\alpha_h) : h \in U_g\}$ and put

$$\mathfrak{B}_{\exp(\alpha_g)} = \{U_{\exp(\alpha_g)} : U_g \in \tau_G\},$$

here τ_G is the topology on the space G. It is easy to check that the family $\mathfrak{B}_{\exp(\alpha_g)}$ forms a neighbourhood system of the point $\exp(\alpha_g) \in \exp(G)$.

Theorem 1. *The set* $\exp(G)$ *is a topological group with respect to the operation* $\exp(\alpha_{g_1})\exp(\alpha_{g_2}) = \exp(\alpha_{g_1 g_2})$. *Moreover,* $\exp(\alpha_e)$ *is a unit of the group* $\exp(G)$ *and* $\exp(\alpha_g)^{-1} = \exp(\alpha_{g^{-1}})$, $g \in G$.

Now for α it is possible to define the action $\alpha^{\exp}\colon \exp(G) \times \exp(X) \to \exp(X)$ by the rule

$$\alpha^{\exp}(\exp(\alpha_g),\, F) = \exp(\alpha_g)(F).$$

Proposition 2. *For the topological transformation groups* $(G,\, X,\, \alpha)$, *the triple* $(\exp(G),\, \exp(X),\, \alpha^{\exp})$ *is a topological transformation groups.*

Proposition 3. *If the set* $A \subset X$ *is* G*-invariant, then the set* $\exp(A)$ *is* $\exp(G)$*-invariant.*

Proposition 4. *For a topological transformation group* (G, X, α), *we have*

$$\ker \alpha^{\exp} = \exp(\ker \alpha).$$

Here $\ker \alpha^{\exp} = \{\exp(\alpha_g) \in \exp(G) : \exp(\alpha_g)(F) = F,\ \forall F \in \exp(X)\}$, $\exp(\ker \alpha) = \{\exp(\alpha_g) \in \exp(G) : g \in \ker \alpha\}$.

Proposition 4 immediately implies

Corollary 5. *The action* $\alpha^{\exp}$ *is effective if and only if the action* α *is effective.*

Note that for the transitive action α of the group G on the space X, the action $\alpha^{\exp}$ induced from α may not be transitive.

Example 6. Let $X = \{x_1,\, x_2,\, x_3\}$ be the discrete topological space (all three points are different). Let

$$G = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \right\}$$

– the discrete topological group of permutations of the index set $\{1,\, 2,\, 3\}$. The action $\alpha\colon G \times X \to X$ of the group G on the space X is defined by the rule $\alpha(g,\, x_i) = x_{g(i)}$. Then α is a transitive action. Moreover, $\alpha_g(x_i) = x_{g(i)}$. It is clear that $\exp(\alpha_g)(\{x_1,\, x_2,\, x_3\}) = \{x_1,\, x_2,\, x_3\}$ for each $g \in G$. Thus, for no closed subset $F \subset X$ there is no element $\exp(\alpha_g)$ of the group $\exp(G)$ for which $\exp(\alpha_g)(F) = \Phi$, here $\Phi = \{x_1,\, x_2,\, x_3\}$, $F \neq \Phi$. Therefore, the action $\alpha^{\exp}$ is not transitive.

Example 6 shows that the action of the group $\exp(G)$ on the space $\exp(X)$ may not be free, although the action of the group G on the space X is free. But, nevertheless, the following is true.

Proposition 7. *Let* $X = \{x_1,\, \ldots,\, x_n\}$ *be a finite discrete space,* G *an arbitrary permutation group (supplied by the discrete topology) of the set* X. *Then, for each free action* α *of the group* G *on the space* X, *the corresponding action* $\alpha^{\exp}$ *of the group* $\exp(G)$ *on the space* $\exp(X)$ *is semi-free. In this case, the only point in the space* $\exp(X)$ *that remains motionless under the action of all elements of* $\exp(G)$ *is the set* $\{x_1,\, \ldots,\, x_n\}$.

It is clear that if G is a compact group, then $\exp(G)$ is also a compact group.

Theorem 8. *The action* $\alpha^{\exp}\colon\ \exp(G)\times\exp(X)\to\exp(X)$ *of the compact group* $\exp(G)$ *on the space* $\exp(X)$ *is a closed map.*

The next statement follows from Theorem 8.

Corollary 9. *If* G *is a compact group and* X *is some* G*-space, then for any closed set* $A\subset\exp(X)$ *the set* $\exp(G)(A)$ *is closed in* $\exp(X)$ *and for compact* A *the set* $\exp(G)(A)$ *is compact.*

Theorem 10. *If* $f\colon\ X\to Y$ *is an equivariant map of one* G*-space to another, then* $\exp(f)\colon\ \exp(X)\to\exp(Y)$ *is also an equivariant map of* $\exp(G)$*-spaces.*

The normality of the functor exp and Theorem 10 imply

Corollary 11. *If* $f\colon X\to Y$ *is an equivalence of* G*-spaces* X *and* Y*, then* $\exp(f)\colon\ \exp X\to\exp Y$ *is an equivalence of* $\exp(G)$*-spaces* $\exp X$ *and* $\exp Y$*.*

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On the Carleman function for matrix factorizations of the Helmholtz equation

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This article deals with the construction of the Carleman function for matrix factorizations of the Helmholtz equation in a multidimensional domain.

It is known that the Cauchy problem for elliptic equations is unstable relatively small change in the data, i.e., is incorrect (Hadamard's example). In unstable problems the image of the operator is not closed, therefore the solvability condition can not be written in terms of continuous linear functionals. Thus, in the Cauchy problem for elliptic equations with data on a part of the boundary of the region, the solution is usually unique, the problem is solvable for an everywhere dense set of data, but this set not closed. Consequently, the theory of solvability of such problems is essentially It is more difficult and deeper than the theory of solvability of the Fredholm equations. The first results in this direction appeared only in the mid-1980s in the works of L.A. Aizenberg, A.M. Kytmanov, N.N. Tarkhanov (See, for instance [1]).

Let $x = (x_1, ..., x_m), y = (y_1, ..., y_m)$ be are points of the Euclidean space $\mathbb{R}^m$ and $G \subset \mathbb{R}^m$ be a bounded simply-connected domain with piecewise smooth boundary consisting of the plane T: $y_m = 0$ and of a smooth surface S lying in the half-space $y_m > 0$, that i.e., $\partial G = S \bigcup T$.

We consider in the domain G a system of differential equations

$$D\left(\frac{\partial}{\partial x}\right) U(x) = 0, \tag{1}$$

where $D\left(\frac{\partial}{\partial x}\right)$ is the matrix of first-order differential operators.

We denote by $A(G)$ the class of vector functions in a domain G continuous on $\overline{G} = G \bigcup \partial G$ and satisfying system (1).

We define the function $\Phi(y, x; \lambda)$ at $y \neq x$ by the following equalities:

$$\Phi(y, x; \lambda) = \frac{1}{c_m K(x_m)} \frac{\partial^{k-1}}{\partial s^{k-1}} \int\limits_0^\infty \operatorname{Im}\left[\frac{K(w)}{w - x_m}\right] \frac{u I_0(\lambda u)}{\sqrt{u^2 + \alpha^2}} du, \; m = 2k, \; k \geq 1, \tag{2}$$

$$\Phi(y, x; \lambda) = \frac{1}{c_m K(x_m)} \frac{\partial^{k-1}}{\partial s^{k-1}} \int\limits_0^\infty \operatorname{Im}\left[\frac{K(w)}{w - x_m}\right] \frac{\cos(\lambda u)}{\sqrt{u^2 + \alpha^2}} du, \; m = 2k+1, \; k \geq 1, \tag{3}$$

where

at $m = 2k$, $k \geq 1$; $c_2 = 2\pi$, $c_m = (-1)^k 2^{-k} (m-2) \pi \omega_m (k-1)!$, $I_0(\lambda u) = J_0(i\lambda u)$–is the Bessel function of the first kind of zero order;

at $m = 2k+1$, $k \geq 1$; $c_m = (-1)^k 2^{-k} (2k-1)! (m-2) \pi \omega_m$, ω_m– the area of a unit sphere in space $\mathbb{R}^m$.

In the future, using formulas (2) and (3), we will construct the Carleman matrix for matrix factorizations of the Helmholtz equation in multidimensional bounded domain and based on it we will find an approximate solution to the Cauchy problem in explicit form, using the methodology of previous works (See, for instance [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17] and [18]).

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Measuring the rate of convergence in the Birkhoff ergodic theorem

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There are two different approaches to the measuring of the rate of convergence in Birkhof's ergodic theorem; see the discussion in [1].

The first one, closely related with probabilities of large deviations, was studied in [2]. Now this approach is well developed. There were obtained estimates of the rate of convergence in the Birkhoff ergodic theorem for many classes of dynamical systems popular in applications, including some well-known billiards and Anosov systems [3].

The second approach (pointwise rate of convergence) was studied in [1] and [4], and many interesting questions still are open here.

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Enumeration of topologically non-equivalent functions with one degenerate saddle critical point on triple torus

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Let $(N, \partial N)$ be a smooth surface with the edge ∂N (∂N can be empty). Let $C^{\infty}(N)$ denote the space of infinitely differentiable functions on N with edge $\partial N = \partial_{-}N \cup \partial_{+}N$, all critical points of which are isolated and lie in the interior of N, and, furthermore, on the connected components of the edge $\partial_{-}N$ ($\partial_{+}N$) the functions from $C^{\infty}(N)$ take the same values a (b accordingly).

Two functions f_1 and f_2 from the space $C^{\infty}(N)$ are called topologically equivalent if there are homeomorphisms $h : N \to N$ and $h' : R^1 \to R^1$ (h' preserves orientation) such that $f_2 = h' \circ f_1 \circ h^{-1}$. If h preserves of the orientation, the functions f_1 and f_2 are called topologically conjugated (eg. [2]) or O-topologically equivalent (eg. [6]).

It is known [2] that a function $f \in C^{\infty}(N)$, in a certain neighborhood of its isolated critical point $x \in N$ (which is not a local extremum) for which the topological type of level lines changes in passing through x, is reduced by a continuous change of coordinates to the form $f = \mathrm{Re}z^n + c$, $n \geq 2$ (are called «essentially» critical point) or $f = \mathrm{Re}z$ if the topological type of level lines does not change in passing through x. The number of essentially critical points x_i of the function f, together with the values of n_i (exponents in there presentation $f = \mathrm{Re}z^{n_i} + c_i$ in the neighborhoods of the critical points x_i), are called the topological singular type of the function f.

The problem of the topological equivalence of functions from the class $C^{\infty}(N)$ with the fixed number of critical points was completely solved by V.V. Sharko in [3] and it was established that a finite number of topologically nonequivalent functions of this class exist. However, it should be noted that the task about calculation of topologically non-equivalent functions with the fixed topological singular type is rather complicated and is still **unsolved**.

When considering functions from the class $C(M_g) \subset C^{\infty}(N)$ that possess **only one** essentially critical point x_0 (degenerate critical point of the saddle type) in addition to local maxima and minima on oriented surface M_g of genus $g \geq 0$, then the problem of counting the number of such non-equivalent functions is greatly simplified. It is well known [2] that $\forall f \in C(M_g)$ the Poincare index of a critical point x_0, is equal $\mathrm{ind}^f(x_0) = 1 - n$, where $n = 2g - 1 + \lambda$ and λ is a total number of local maxima and minima.

Let $C_n(M_g)$ be a class of functions from $C(M_g)$ which, in addition to local maxima and minima, have only one essentially critical point, whose the Poincare index is equal $(1-n)$. Denote the class of functions from $C(M_g)$ that possess one essentially critical point, k local maxima and l local minima by $C_{k,l}(M_g)$. Then it is clear that $\forall f \in C_{k,l}(M_g)$ $n = 2g - 1 + k + l$.

In the general case, for natural g, k, l (or k, l, and $n = 2g + k + l - 1$), the problem of calculating the number of topologically non-equivalent functions from the class $C_{k,l}(M_g)$ also has proved to be quite a difficult and unsolved problem to date.

The task of calculating the number of topologically non-equivalent functions from the class $C_{1,1}(M_g)$ (for genus $g \geq 1$) was completely solved in [4]. In [5], for natural k and l solved completely the problems of calculating the numbers O-topologically and topologically non-equivalent functions from the class $C_{k,l}(M_0)$ (on two-dimensional sphere).

In [6], [7] solved completely the problems of calculating the numbers O-topologically and topologically non-equivalent functions from the class $C_{1,n-2}(M_1)$ and $C_{1,n-4}(M_2)$ accordingly.

In general case, for functions from the class $C_{1,n-2g}(M_g)$, the task is also still unsolved.

By using the results of [1] we can establish the validity of the following statement

Theorem 1. *For natural $n \geq 7$ the number $d^*(n)$ of O-topologically non-equivalent functions from the class $C_{1,n-6}(M_3)$ can be calculated by the relation*

$$d^*(n) = \frac{1}{n}\left(d(n) + \sum_{j|n,\ j\in\{2;3;4;6;7;8;9;12\}} \phi(j)\cdot\rho\left(n, \tfrac{n}{j}\right)\right), \tag{1}$$

where: $\phi(q)$ *is the Euler totient function;*

$$d(n) = \frac{1}{72}C_{n+1}^{8}\cdot(9n^4 - 18n^3 - 57n^2 + 34n + 80); \tag{2}$$

$\forall j \in N : \frac{n}{j} \notin N$ *the value* $\rho\left(n, \frac{n}{j}\right) \equiv 0;$ $\forall j \in \{2; 3; 4; 6; 7; 8; 9; 12\} : \frac{n}{j} \in N$ *the value of* $\rho\left(n, \frac{n}{j}\right)$ *can be calculated by the relations*

$$\rho\left(n, \tfrac{n}{12}\right) = \tfrac{n}{12},\quad \rho\left(n, \tfrac{n}{9}\right) = \tfrac{2n}{9},\quad \rho\left(n, \tfrac{n}{8}\right) = \tfrac{n}{4},\quad \rho\left(n, \tfrac{n}{7}\right) = \tfrac{5n}{7},\quad \rho\left(n, \tfrac{n}{4}\right) = \tfrac{n(n-4)(n+40)}{384},$$

$$\rho\left(n, \tfrac{n}{6}\right) = \tfrac{n(n-6)}{72},\quad \rho\left(n, \tfrac{n}{3}\right) = \tfrac{n(n-3)(n-6)(3n+29)}{648},\quad \rho\left(n, \tfrac{n}{2}\right) = \tfrac{n(n-2)(n-4)(n-6)(37n^2+294n-2320)}{46080}. \tag{3}$$

n	$d(n)$	$d^*(n)$	n	$d(n)$	$d^*(n)$
7	180	30	**19**	1 801 329 010	94 806 790
8	3 044	385	**20**	3 600 529 450	180 028 084
9	26 060	2 900	**21**	6 925 187 830	329 770 930
10	152 900	15 308	**22**	12 869 925 310	584 999 362
11	696 905	63 355	**23**	23 190 544 696	1 008 284 552
12	2 641 925	220 242	**24**	40 637 416 600	1 693 230 295
13	8 691 683	668 591	**25**	69 427 501 000	2 777 100 040
14	25 537 655	1 824 311	**26**	115 901 728 800	4 457 765 752
15	68 396 900	4 559 818	**27**	189 426 912 675	7 015 811 753
16	169 537 940	10 596 558	**28**	303 616 322 295	10 843 450 498
17	393 481 660	23 145 980	**29**	477 960 911 025	16 481 410 725
18	862 928 092	47 941 370	**30**	739 984 318 125	24 666 159 267

TABLE 1.1. Number $d^*(n)$ of O-topologically non-equivalent functions from the class $C_{1,n-6}(M_3)$

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On some properties of moduli of smoothness of conformal mapping of simply connected domains

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Let consider a function realizing homeomorphism of the closed unit disk onto the closure of simply connected domain in the complex plane bounded by a smooth Jordan curve conformal in the open unit disk. Let suppose that boundary of this domain is characterized by the angle between the tangent to the curve and the positive real axis considered as the function of the arc length on the curve.

O. D. Kellogg in 1912 proved the theorem in which it had been established that if this angle function satisfies Holder condition, then the derivative of the function realizing mapping of unit disk onto the closure of the given domain satisfies Holder condition with the same index. Connection between properties of the boundary of the domain and properties of the considered function was investigated in works by numerous authors: S. E. Warshawski, J. L. Geronimus, S. J. Alper, R. N. Kovalchuk, L. I. Kolesnik, P. M. Tamrazov (more detailed see [1–3], [5] and [6]). Some close problems were investigated by V. A. Danilov, E. P. Dolzenko, E. M. Dynkin, N. A. Shirokov, S. R. Bell and S. G. Krantz, V. V. Andrievskii, V. I. Belyi, B. Oktay, D. M. Israfilov and others (more detailed see [3–5] and [7]).

Certain results in terms of moduli of smoothness of different types (uniform curvilinear, arithmetic, local and integral moduli of smoothness of arbitrary order) were received by author. In particular, some estimates for integral moduli of smoothness were considered in [4–6].

Let $\omega_{k,z}(f(z),\delta)$ be a noncentralized local arithmetic modulus of smoothness of order k ($k \in \mathbb{N}$) of the function $w = f(z)$ at a point z on the curve γ [1]. Let consider the integral modulus of smoothness of order k for the function $w = f(z)$ on the curve γ introduced in [2] by the formula $\widehat{\omega}_k(f(z),\delta) = \left(\int\limits_{\gamma} [\omega_{k,z}(f(z),\delta)]^p \, d\lambda(z) \right\}^{1/p}$, $1 \le p < +\infty$, $k \in \mathbb{N}$, where $\lambda = \lambda(z)$ is the linear Lebesgue's measure on the curve.

Let G_1 and G_2 be the simply connected domains in the complex plane bounded by the smooth Jordan curves Γ_1 and Γ_2. Let $\tau_1(s_1)$ be the angle between the tangent to Γ_1 and the positive real axis, $s_1(z)$ be the arc length on Γ_1. Let $\tau_2(s_2)$ be the angle between the tangent to Γ_2 and the positive real axis, $s_2(w)$ be the arc length on Γ_2. Let $w = f(z)$ be a homeomorphism of the closure $\overline{G_1}$ of the domain G_1 onto the closure $\overline{G_2}$ of the domain G_2, conformal in open domain G_1.

Theorem 1 ([5]). *If integral moduli of smoothness $\widehat{\omega}_k(\tau_1(s_1),\delta)$ and $\widehat{\omega}_k(\tau_2(s_2),\delta)$ of order k ($k \in \mathbb{N}$) for the functions $\tau_1(s_1)$ and $\tau_2(s_2)$ satisfy Holder condition $\widehat{\omega}_k(\tau_1(s_1),\delta) = O(\delta^\alpha)$ $(\delta \to 0)$ and $\widehat{\omega}_k(\tau_2(s_2),\delta) = O(\delta^\alpha)$ $(\delta \to 0)$, with the same index α, $0 < \alpha < k$, then integral modulus of smoothness $\widehat{\omega}_k(f'(z),\delta)$ of the derivative of the function $f(z)$ on Γ_1 satisfies Holder condition $\widehat{\omega}_k(f'(z),\delta) = O(\delta^\alpha)$ $(\delta \to 0)$ with the same index α.*

Theorem 2 ([6]). *If integral moduli of smoothness $\widehat{\omega}_k(\tau_1(s_1),\delta)$ and $\widehat{\omega}_k(\tau_2(s_2),\delta)$ of order k ($k \in \mathbb{N}$) for the functions $\tau_1(s_1)$ and $\tau_2(s_2)$ satisfy condition $\widehat{\omega}_k(\tau_1(s_1),\delta) = O(\delta^k \log\frac{1}{\delta})$ $(\delta \to 0)$ and $\widehat{\omega}_k(\tau_2(s_2),\delta) = O(\delta^k \log\frac{1}{\delta})$ $(\delta \to 0)$, then integral modulus of smoothness $\widehat{\omega}_k(f'(z),\delta)$ of the derivative of the function $f(z)$ on Γ_1 satisfies condition $\widehat{\omega}_k(f'(z),\delta) = O(\delta^k \log\frac{1}{\delta})$ $(\delta \to 0)$.*

Theorem 3. *Let integral moduli of smoothness $\widehat{\omega}_k(\tau_1(s_1),\delta)$ and $\widehat{\omega}_k(\tau_2(s_2),\delta)$ of order k ($k \in \mathbb{N}$) for the functions $\tau_1(s_1)$ and $\tau_2(s_2)$ satisfy conditions $\widehat{\omega}_k(\tau_1(s_1),\delta) = O(\omega(\delta))$ $(\delta \to 0)$ and*

$\widehat{\omega}_k(\tau_2(s_2), \delta) = O(\omega(\delta))$ $(\delta \to 0)$, *where* $\omega(\delta)$ *is normal majorant satisfying the condition* $\int\limits_0^l \frac{\omega(t)}{t} dt < +\infty$. *Then integral modulus of smoothness* $\widehat{\omega}_k(f(z), \delta)$ *of the function* $f(z)$ *on* Γ_1 *satisfies the condition* $\widehat{\omega}_k(f(z), \delta) = O(\sigma(\delta))$ $(\delta \to 0)$, *where* $\sigma(\delta)$ *is some integral majorant.*

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Integration over non-rectifiable curves: spirals of high torsion

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The presentation is devoted to some new results related to the classical problem of complex analysis - the Riemann boundary value problem. We, however, consider it on a non-rectifiable curve and pay considerable attention to the geometric features of the curve. In this case, we will talk about spirals, which we will classify depending on the speed of it's twisting. We consider the geometric properties of arcs in the neighbourhood of their ends.

Let us consider first a well known boundary-value problem of complex analysis – so called Riemann problem on a simple Jordan arc (see, for instance, [1, 6, 3]).

Given a directed Jordan arc Γ in the complex plane $\mathbb{C}$ with beginning and end at points a_1 and a_2 relatively, and two functions $G(t)$, $g(t)$, $t \in \Gamma$. Find all holomorphic in $\overline{\mathbb{C}} \setminus \Gamma$ functions $\Phi(z)$ which vanish at ∞ and have boundary limits $\Phi^{\pm}(t)$ from the left and from the right correspondingly at any point $t \in \Gamma \setminus \{a_1, a_2\}$ such that

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t), \quad t \in \Gamma \setminus \{a_1, a_2\}. \tag{1}$$

In addition, the desired function Φ must satisfy certain conditions on its growth at the end points $a_{1,2}$.

In numerous classical works (see [1, 6, 3] and many other) the solutions of this problem are obtained in terms of Cauchy type integrals. Particularly, a solution of so called jump problem

$$\Phi^+(t) = \Phi^-(t) + g(t), \quad t \in \Gamma \setminus \{a_1, a_2\}, \tag{2}$$

on piecewise - smooth arc Γ is representable as the Cauchy type integral

$$\Phi(z) = \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{g(t)\,dt}{t - z}, \quad z \notin \Gamma. \tag{3}$$

As a result, in all classical works on this problem the boundary Γ is assumed rectifiable, although the formulation of the problem does not imply this restriction. It keeps the sense for non-rectifiable arcs, too. The Riemann boundary-value problem for non-rectifiable boundaries was solved first in the papers [4, 5, 6, 7].

We introduce the concept of torsion of arc Γ [8]. The torsion of arc Γ at point a_j, $j = 1, 2$ is a value

$$\tau_j := \inf \left\{ p > 0 : \iint |K_\Gamma(z)|^{1/p} dx\, dy < \infty \right\},$$

where integral is taken over a neighborhood of a_j. If $\tau_j < 1$, then we say that the arc has moderate torsion at point a_j, otherwise its torsion is high. This value characterizes the rate of curling of Γ around the point a_j.

As it appears, the torsion concept is very closely connected with the integrator concept that we introduced earlier - and this allows us to get some new results in this geometric terms.

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On real Σ^*-algebras of operators

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Real Σ^*-algebras of operators are introduced and their connections with (complex) Σ^*-algebras and real von Neumann algebras are discussed.

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On controllability problems for the heat equation with variable coefficients controlled by the Dirichlet boundary condition on a half-axis

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Consider the heat equation

$$w_t = \frac{1}{\rho}\left(k w_x\right)_x + \gamma w, \qquad x \in (0,+\infty),\ t \in (0,T), \tag{1}$$

controlled by the Dirichlet boundary condition

$$w(0,\cdot) = u, \qquad t \in (0,T), \tag{2}$$

under the initial condition

$$w(\cdot,0) = w^0, \qquad x \in (0,+\infty), \tag{3}$$

where $T = \text{const} > 0$; ρ, k, γ, w^0 are given functions; $u \in L^\infty(0,T)$ is a control. We assume $\rho, k \in C^1[0,+\infty)$, $\rho > 0$ and $k > 0$ on $[0,+\infty)$, $(\rho k) \in C^2[0,+\infty)$, $(\rho k)'(0) = 0$, and

$$\sigma(x) = \int_0^x \sqrt{\rho(\mu)/k(\mu)}\, d\mu \to +\infty \qquad \text{as } x \to +\infty.$$

In addition, we assume

$$(P(k,\rho) - \gamma) \in L^\infty(0,+\infty) \bigcap C^1[0,+\infty) \qquad \text{and} \qquad \sigma\sqrt{\frac{\rho}{k}}\,(P(k,\rho) - \gamma) \in L^1(0,+\infty),$$

where $P(k,\rho) = \frac{1}{4}\sqrt{\frac{k}{\rho}}\left(\sqrt{\frac{k}{\rho}}\left(\frac{k'}{k} + \frac{\rho'}{\rho}\right)\right)' + \left(\frac{1}{4}\sqrt{\frac{k}{\rho}}\left(\frac{k'}{k} + \frac{\rho'}{\rho}\right)\right)^2$.

Control system (1)–(3) is considered in modified Sobolev spaces. Denote $\eta = (k\rho)^{1/4}$, $\theta = \left(\frac{k}{\rho}\right)^{1/4}$, $\mathcal{D}_{\eta\theta} = \theta^2\left(\frac{d}{dx} + \frac{\eta'}{\eta}\right)$. Denote also $\mathbb{R}_+ = (0,+\infty)$. For $p = 1,2$, denote

$$\widetilde{\mathbb{H}}^0_+ = \left\{\varphi \in L^2_{\mathrm{loc}}(\mathbb{R}_+) \mid \left(\frac{\eta}{\theta}\varphi\right) \in L^2(\mathbb{R}_+)\right\}, \quad \widetilde{\mathbb{H}}^p_+ = \left\{\varphi \in \widetilde{\mathbb{H}}^{p-1}_+ \mid \left(\frac{\eta}{\theta}\mathcal{D}^p_{\eta\theta}\varphi\right) \in L^2(\mathbb{R}_+) \text{ and } \varphi(0) = 0\right\},$$

with the norm

$$[\![\varphi]\!]^s_+ = \sqrt{\sum_{m=0}^{s}\left(\left\|\frac{\eta}{\theta}\left(\mathcal{D}^m_{\eta\theta}\varphi\right)\right\|_{L^2(\mathbb{R}_+)}\right)^2}, \qquad \varphi \in \widetilde{\mathbb{H}}^s_+, \quad s = 0,1,2,$$

and the dual spaces $\widetilde{\mathbb{H}}^{-s}_+ = \left(\widetilde{\mathbb{H}}^s_+\right)^*$ with the norms associated with the strong topology of these spaces, $s = 0,1,2$.

We suppose $\left(\frac{d}{dt}\right)^p w : [0,T] \to \widetilde{\mathbb{H}}^{-2p}_+$, $p = 0,1$, and $w^0 \in \widetilde{\mathbb{H}}^0_+$ in system (1)–(3).

Let $T > 0$ and $w^0 \in \widetilde{\mathbb{H}}^0_+$. By $\mathcal{R}_T(w^0)$, denote the set of all states $w^T \in \widetilde{\mathbb{H}}^0_+$ for which there exists a control $u \in L^\infty(0,T)$ such that for the solution w to system (1)–(3) we have $w(\cdot,T) = w^T$.

Definition 1. A state $w^0 \in \widetilde{\mathbb{H}}^0_+$ is said to be controllable to a state $w^T \in \widetilde{\mathbb{H}}^0_+$ with respect to system (1)–(3) in a given time $T > 0$ if $w^T \in \mathcal{R}_T(w^0)$.

Definition 2. A state $w^0 \in \widetilde{\mathbb{H}}^0_+$ is said to be null-controllable with respect to system (1)–(3) in a given time $T > 0$ if $0 \in \mathcal{R}_T(w^0)$.

Definition 3. A state $w^0 \in \widetilde{\mathbb{H}}^0_+$ is said to be approximately controllable to a state $w^T \in \widetilde{\mathbb{H}}^0_+$ with respect to system (1)–(3) in a given time $T > 0$ if $w^T \in \overline{\mathcal{R}_T(w^0)}$, where the closure is considered in the space $\widetilde{\mathbb{H}}^0_+$.

Consider also control system with the simplest heat operator (the case $\rho = k = 1$, $\gamma = 0$)

$$z_t = z_{yy}, \qquad y \in (0,+\infty),\ t \in (0,T), \tag{4}$$

$$z(0,\cdot) = v, \qquad t \in (0,T), \tag{5}$$

$$z(\cdot,0) = z^0, \qquad y \in (0,+\infty), \tag{6}$$

where $v \in L^\infty(0,T)$ is a control, $\left(\frac{d}{dt}\right)^m z : [0,T] \to \widetilde{H}^{-2m}_+$, $m = 0,1$, $w^0 \in \widetilde{H}^0_+ = L^2(\mathbb{R}_+)$. Here $\widetilde{H}^s_+ = \left\{\varphi \in L^2(\mathbb{R}_+) \mid \left(\forall k = \overline{0,s}\ \varphi^{(k)} \in L^2(\mathbb{R}_+)\right) \text{ and } \left(\forall k = \overline{0,s-1}\ \varphi^{(k)}(0^+) = 0\right)\right\}$, $\quad s = 0,1,2,$ with the norm

$$\|\varphi\|^s_+ = \sqrt{\sum_{k=0}^{s} \binom{s}{k} \left(\left\|\varphi^{(k)}\right\|_{L^2(\mathbb{R}_+)}\right)^2}, \qquad \varphi \in \widetilde{H}^s_+, \quad s = 0,1,2,$$

and $\widetilde{H}^{-s}_+ = \left(\widetilde{H}^s_+\right)^*$ with the norms associated with the strong topology of these spaces, $s = 0,1,2$.

Controllability problems for system (4)–(6) were investigated in [1].

To study controllability problems for system (1)–(3), we use the transformation operator $\widetilde{\mathbb{T}} : \widetilde{H}^{-2}_+ \to \widetilde{\mathbb{H}}^{-2}_+$. It was introduced and studied in [2]. In particular, it has been proved therein that $\widetilde{\mathbb{T}}$ is a continuous one-to-one mapping between the spaces $\widetilde{H}^s$ and $\widetilde{\mathbb{H}}^s$, $s = -2,-1,0$.

In the present talk, we prove that the transformation operator $\widetilde{\mathbb{T}}$ is one-to-one mapping between the sets of the solutions to system (4)–(6) and to system (1)–(3). The application of the operator $\widetilde{\mathbb{T}}$ allows us to conclude that the control system (1)–(3) replicates the controllability properties of the control system (4)–(6) and vice versa. A relation between controls u and v is also found. Thus, we obtain the following main results.

Theorem 4. *If a state $w^0 \in \widetilde{\mathbb{H}}^0_+$ is null-controllable with respect to system (1)–(3) in a time $T > 0$, then $w^0 = 0$.*

Theorem 5. *Each state $w^0 \in \widetilde{\mathbb{H}}^0_+$ is approximately controllable to any target state $w^T \in \widetilde{\mathbb{H}}^0_+$ with respect to system (1)–(3) in a given time $T > 0$.*

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Perfect metrizability of the functor of idempotent measures

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Let $\mathbb{R}$ be the real line. The set $\mathbb{R}\cup\{-\infty\}$ considered with operations: addition $\oplus$ and multiplication $\odot$ defined as $u\oplus v=\max\{u,\,v\}$ and $u\odot v=u+v$, denotes by $\mathbb{R}_{\max}$. Let X be a compact Hausdorff space, $C(X)$ the algebra of continuous functions $\varphi\colon X\to\mathbb{R}$ with the usual algebraic operations. On $C(X)$ the operations $\oplus$ and $\odot$ we define as $\varphi\oplus\psi=\max\{\varphi,\,\psi\}$, $\varphi\odot\psi=\varphi+\psi$, $\lambda\odot\varphi=\varphi+\lambda_X$ here $\varphi,\,\psi\in C(X)$, $\lambda\in\mathbb{R}$. Recall [1] that a functional $\mu\colon C(X)\to\mathbb{R}$ is said to be an *idempotent probability measure* on X, if: 1) $\mu(\lambda_X)=\lambda$ for each $\lambda\in\mathbb{R}$; 2) $\mu(\lambda\odot\varphi)=\mu(\varphi)+\lambda$ for all $\lambda\in\mathbb{R}$, $\varphi\in C(X)$; 3) $\mu(\varphi\oplus\psi)=\mu(\varphi)\oplus\mu(\psi)$ for every $\varphi,\,\psi\in C(X)$. The set of all idempotent probability measures on X we denote by $I(X)$. Consider $I(X)$ as a subspace of $\mathbb{R}^{C(X)}$. The topological space $I(X)$ is compact [1]. For a given map $f\colon X\to Y$ of compact Hausdorff spaces the map $I(f)\colon I(X)\to I(Y)$ defines by the formula $I(f)(\mu)(\varphi)=\mu(\varphi\circ f)$, $\mu\in I(X)$, where $\varphi\in C(Y)$. The construction I is a normal covariant functor, acting in the category of compact Hausdorff spaces and their continuous maps. For $\mu\in I(X)$ we may define the support of μ : $\operatorname{supp}\mu=\cap\{A\subset X:\,\bar{A}=A,\,\mu\in I(A)\}$. For a point $x\in X$ by the rule $\delta_x(\varphi)=\varphi(x)$, $\varphi\in C(X)$, we define the Dirac measure δ_x supported on the singleton $\{x\}$.

Put

$$U_S(X)=\big\{\lambda\colon X\to[-\infty,\,0]\,\big|\quad \lambda \text{ is upper semicontinuous and there exists a } x_0\in X \text{ such that } \lambda(x_0)=0\big\}.$$

Then we have

$$I(X)=\left\{\bigoplus_{x\in X}\lambda(x)\odot\delta_x\,:\,\lambda\in U_S(X)\right\}.$$

We define a subset

$$I_\omega(X)=\left\{\bigoplus_{x\in X}\lambda(x)\odot\delta_x\,:\,\lambda\in U_S(X),\,|\{x\in X\,:\,\lambda(x)>-\infty\}|<\infty\right\}\subset I(X).$$

$I_\omega(X)$ is everywhere dense in $I(X)$ [1, 2]. Put

$$\rho_2(\mu_1,\,\mu_2)=\inf\left\{\frac{\sum\limits_{(x,\,y)\in\operatorname{supp}\xi}\mathrm{e}^{\lambda_1(x)+\lambda_2(y)}\cdot\rho(x,\,y)}{\sum\limits_{x\in\operatorname{supp}\mu_1}\mathrm{e}^{\lambda_1(x)}\cdot\sum\limits_{y\in\operatorname{supp}\mu_2}\mathrm{e}^{\lambda_2(y)}}\,:\,\xi\in\Lambda_{1\,2}\right\},$$

where $\mu_i=\bigoplus\limits_{x\in X}\lambda_i(x)\odot\delta_x\in I_\omega(X)$, $i=1,\,2$. Further, for every pair $\mu,\,\nu\in I(X)$ take consequences $\{\mu_n\}$, $\{\nu_n\}\subset I_\omega(X)$ such that $\lim\limits_{n\to\infty}\mu_n=\mu$ and $\lim\limits_{n\to\infty}\nu_n=\nu$, and put

$$\rho_I(\mu,\,\nu)=\lim_{n\to\infty}\rho_2(\mu_n,\nu_n).$$

The function ρ_I is a metric on $I(X)$ generating the pointwise convergence topology on $I(X)$ and the restriction of which coincides with the metric ρ on X.

Consider a system ψ consisting of all mapgs $\psi_X\colon\ I^2(X)\to I(X)$, acting as the following. Given $M\in I^2(X)$ put $\psi_X(M)(\varphi)=M(\overline{\varphi})$, where for any function $\varphi\in C(X)$ the function $\overline{\varphi}\colon I(X)\to\mathbb{R}$ defines by the formula $\overline{\varphi}(\mu)=\mu(\varphi)$. Fix a compactum X and for a positive integer n put $\psi_{n+1,\,n}=\psi_{I^{n-1}(X)}\colon\ I^{n+1}(X)\to I^n(X)$. Note that $\psi_{n+1,\,n}\circ\eta_{n,\,n+1}=Id_{I^n(X)}$.

Lemma 1. $\psi_{1,0}\colon (I^2(X),\, \rho_{I^2}) \to (I(X),\, \rho_I)$ is a non-expanding map.

Lemma 2. For each $N \in \psi_{1,0}^{-1}(\mu)$ we have $\rho_I(\mu,\, \delta_{x_0}) = \rho_{I^2}(\delta_{\delta_{x_0}},\, N)$.

Lemma 3. If $\rho_I(\mu,\, \eta_{0,1}(X)) \geq \varepsilon$ then $\rho_{I^2}(I(\eta_{0,1})(\mu),\; \eta_{1,2}(I(X))) \geq \varepsilon$.

Theorem 4. The functor I is perfect metrizable.

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Quasiareal deformation of surfaces of positive Gauss curvature

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In this paper it is considered quasiareal deformation of surfaces, which we will call also briefly QA-deformation. Quasiareal deformation is understood as an infinitesimal deformation of the first order with the given law of changing the element of area of a surface in Euclidean three-space.

Let $\overline{U}\left(x^1, x^2\right)$ be a field of velocities of the points of the surface $\overline{r} = \overline{r}\left(x^1, x^2\right)$ at the initial moment of the deformation, such that $\overline{U} = U^\alpha \overline{r}_\alpha + U^0 \overline{n}$, where $\overline{r_i}, \overline{n}$, $i = 1, 2$, are the basis vectors. The fundamental equations of the quasiareal infinitesimal deformation, which are expressed in terms of the components of the partial derivatives of the field $\overline{U}$, are derived in [2].

It has been established: in order that the field $\overline{U} \in C^1$ be a deforming field of the quasiareal infinitesimal deformation it is necessary and sufficient that the components U^α, U^0 satisfy the equation

$$U^\alpha_{,\alpha} - 2HU^0 = -2\mu, \tag{1}$$

where the function μ expresses the law of changing the element of area.

It is evident, that the class of the QA-deformation is very wide since one differential equation (1) contains four unknown functions. It is expedient to study such deformation under the additional geometrical or mechanical conditions. For example, for the surface of positive Gauss curvature ($K > 0$) on the condition that $\delta\overline{n} = 0$ under the quasiareal infinitesimal deformation we have additional elliptic partial differential equation of the second order with respect to the normal component of the deforming field

$$d^{\alpha\beta} U^0_{\alpha,\beta} - \frac{K_\alpha}{K} d^{\alpha\beta} U^0_\beta + 2HU^0 = 2\mu. \tag{2}$$

The Riemann domain T has been described, in which the regular solution of the equation (2) exists for the regular surfaces of positive Gauss curvature, this solution is a continuous, non-zero everywhere in closed domain $\overline{T}$. This condition is a sufficient sign of the existence and uniqueness of the solution of the Dirichlet problem for the equation (2) [1].

The corresponding theorems have been formulated for the QA-deformation of the surfaces of positive Gauss curvature. QA-deformation in class of surfaces of constant mean curvature is discussed, for example, in a paper [3] and deformations preserving Gauss curvature in a paper [4].

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Canonical infinitesimal deformations of metrics of pseudo-Riemannian spaces

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Difference of metric tensors of two pseudo-Riemannian spaces is called their deformation. Let V_n — be a pseudo-Rimeannian space with a metric tensor g_{ij}, and $\bar{V}_n$ — a pseudo-Riemannian space with a metric tensor $\bar{g}_{ij}$. Let us suppose that metric tensors differ by an infinitesimal small number γ_{ij}, or

$$\bar{g}_{ij} = g_{ij} + \gamma_{ij}. \tag{1}$$

Infinitesimal small numbers with an order above the first one will be discarded. Then, the following expression is true for tensors that are reversed in respect to metric tensors.

$$\bar{g}^{ij} = g^{ij} - g^{i\alpha} g^{j\beta} \gamma_{\alpha\beta}.$$

Components of tensor γ_{ij} are called components of the tensor field of velocities of infinitesimal small deformation.

While calculating other inner geometric objects, there is often a need to discard certain parameters. This way leads to the research on infinitesimal deformations of a metric. In this sense, infinitesimal parameters are parameters, which can be discarded not affecting the completeness of the problem under study.

Infinitesimally small deformation of type (1) of pseudo-Riemannian space (V_n, g_{ij}) is called canonical deformation when deformation tensor δg_{ij} can be represented in a shape

$$\gamma_{ij} = \overset{1}{\tau} g_{ij} + \overset{2}{\tau} R_{ij},$$

where $\overset{1}{\tau}, \overset{2}{\tau}$ — are some invariants [1, 2].

Since Saint-Venant's times, the deformation research is reduced to analysis of a system of differential equations. Saint-Venant's equations are the main tool for research on infinitesimal deformations. Saint-Venant's equations are understood here as a set of equations defining the deformation tensor in such a way that the space remains an Euclidean space.

Generalized Saint-Venant's equations are conditions under which Riemann tensor is preserved under infinitesimal deformations of a metric tensor of a pseudo-Riemannian space. They are differential equations in covariant derivatives in respect to tensors of Ricci and Riemann.

Conditions, which are imposed on tensors used for research on infinitesimal deformations, are both algebraic and differential. Having carried out the research of this type we are able to answer the question: whether the Saint-Venant's equations are true under the pre-defined conditions.

The research is carried out locally in tensor form, without limitations on a sign of a metric tensor.

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On the behavior at infinity of ring Q-homeomorphisms

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Let Γ be a family of curves γ in $\mathbb{R}^n$, $n \geqslant 2$. A Borel measurable function $\rho : \mathbb{R}^n \to [0, \infty]$ is called *admissible* for Γ, (abbr. $\rho \in \operatorname{adm}\Gamma$), if

$$\int\limits_{\gamma} \rho(x)\, ds \ \geqslant\ 1$$

for any curve $\gamma \in \Gamma$. Let $p \in (1, \infty)$. The quantity

$$M_p(\Gamma) \ =\ \inf_{\rho \in \operatorname{adm}\Gamma} \int\limits_{\mathbb{R}^n} \rho^p(x)\, dm(x)$$

is called *p–modulus* of the family Γ.

For arbitrary sets E, F and G of $\mathbb{R}^n$ we denote by $\Delta(E, F, G)$ a set of all continuous curves $\gamma : [a, b] \to \mathbb{R}^n$, that connect E and F in G, i.e., such that $\gamma(a) \in E$, $\gamma(b) \in F$ and $\gamma(t) \in G$ for $a < t < b$.

Let D be a domain in $\mathbb{R}^n$, $n \geqslant 2$, $x_0 \in D$ and $d_0 = \operatorname{dist}(x_0, \partial D)$. Set

$$\mathbb{A}(x_0, r_1, r_2) = \{x \in \mathbb{R}^n : r_1 < |x - x_0| < r_2\}\,,$$

$$S_i = S(x_0, r_i) = \{x \in \mathbb{R}^n : \ |x - x_0| = r_i\}\,, \quad i = 1, 2\,.$$

Let a function $Q : D \to [0, \infty]$ be Lebesgue measurable. We say that a homeomorphism $f : D \to \mathbb{R}^n$ is ring Q-homeomorphism with respect to p-modulus at $x_0 \in D$, if the relation

$$M_p(\Delta(fS_1, fS_2, fD)) \ \leqslant\ \int\limits_{\mathbb{A}} Q(x)\, \eta^p(|x - x_0|)\, dm(x)$$

holds for any ring $\mathbb{A} = \mathbb{A}(x_0, r_1, r_2)$, $0 < r_1 < r_2 < d_0$, $d_0 = \operatorname{dist}(x_0, \partial D)$, and for any measurable function $\eta : (r_1, r_2) \to [0, \infty]$ such that

$$\int\limits_{r_1}^{r_2} \eta(r)\, dr = 1\,.$$

Denote by ω_{n-1} the area of the unit sphere $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ in $\mathbb{R}^n$ and by $q_{x_0}(r) = \frac{1}{\omega_{n-1} r^{n-1}} \int\limits_{S(x_0,r)} Q(x)\, d\mathcal{A}$ the integral mean over the sphere $S(x_0,r) = \{x \in \mathbb{R}^n : |x - x_0| = r\}$, here $d\mathcal{A}$ is the element of the surface area. Let $L(x_0, f, R) = \sup\limits_{|x-x_0|\leqslant R} |f(x) - f(x_0)|$.

Theorem. *Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is a ring Q-homeomorphism with respect to p-modulus at a point x_0 with $p > n$ where x_0 is some point in $\mathbb{R}^n$. Then for all numbers $r_0 > 0$ the estimate*

$$\varliminf_{R\to\infty} \left(L(x_0, f, R) \left(\int\limits_{r_0}^{R} \frac{dt}{t^{\frac{n-1}{p-1}}\, q_{x_0}^{\frac{1}{p-1}}(t)} \right)^{-\frac{p-1}{p-n}} \right) \geqslant \left(\frac{p-n}{p-1} \right)^{\frac{p-1}{p-n}} > 0$$

holds.

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Elements of probability theory and measures with values in hypercomplex algebras

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In recent years, the expansion of probability theory and measure theory from real values to values in hypercomplex numbers are actively studied because of their possible applications in mathematics and physics [1] – [5]. In this paper, we extend the notion of probability measure to the case where the measure takes values in the algebra of bihyperbolic numbers [6]. In addition, the concept of the real-valued measure is generalized to the quaternionic-valued measure [7].

The bihyperbolic numbers forms a 4-dimensional algebra over the field of real numbers $\mathbb{W}_4 = \{a_0 + a_1 e + a_2 f + a_3 g,\ a_i \in \mathbb{R},\ i = 0, 1, 2, 3\}$ with basis $\{1, e, f, g\}$ and the following multiplications $e^2 = f^2 = g^2 = 1,\ \ ef = fe = g,\ eg = ge = f,\ fg = gf = e$.

Lemma 1. [8] *Any bihyperbolic number α can be represented as $\alpha = r_1 i_1 + r_2 i_2 + r_3 i_3 + r_4 i_4$, where i_k are idempotents of algebra $\mathbb{W}_4$, $r_k \in \mathbb{R}$, $k = 1, 2, 3, 4$.*

We define on $\mathbb{W}_4$ the relation of partial order $\preccurlyeq$ such as $\alpha \preccurlyeq \beta \Longleftrightarrow \beta - \alpha \in \mathbb{W}_4^+ = \{x_1 i_1 + x_2 i_2 + x_3 i_3 + x_4 i_4 |\ x_k \geq 0,\ k = 1, 2, 3, 4\}$. If $\alpha \preccurlyeq \beta$ but $\alpha \neq \beta$, we denote $\alpha \prec \beta$. Let us denote by A_x, the set of all bihyperbolic numbers which are not $\mathbb{W}_4$-comparable with $x \in \mathbb{W}_4$.

Definition 2. The $\mathbb{W}_4$-valued modulus of a bihyperbolic number $\alpha = r_1 i_1 + r_2 i_2 + r_3 i_3 + r_4 i_4$ is said to be $|\alpha|_{\mathbb{W}_4} = |r_1 i_1 + r_2 i_2 + r_3 i_3 + r_4 i_4|_{\mathbb{W}_4} = |r_1|\, i_1 + |r_2|\, i_2 + |r_3|\, i_3 + |r_4|\, i_4 \in \mathbb{W}_4^+$, where $|r_1|$, $|r_2|$, $|r_3|$, $|r_4|$ are ordinary modules of real numbers.

Definition 3. Let (Ω, Σ) be a measurable space. The function $P_{\mathbb{W}_4} : \Sigma \to \mathbb{W}_4$ is called a $\mathbb{W}_4$-valued probability (or bihyperbolic probability) on the σ-algebra of events Σ, if the following conditions hold: 1) $P_{\mathbb{W}_4}(A) \succcurlyeq 0, \forall A \in \Sigma$; 2) $P_{\mathbb{W}_4}(\Omega) = \zeta$, where $\zeta = 1, i_1, i_2, i_3, i_4$; 3) For any sequence $\{A_n,\ n \geq 1\} \subset \Sigma$ of pairwise incompatible random events we have $P_{\mathbb{W}_4}\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P_{\mathbb{W}_4}(A_n)$.

The triplet $(\Omega, \Sigma, P_{\mathbb{W}_4})$ is called a $\mathbb{W}_4$-probability space.

Each $\mathbb{W}_4$-valued probability measure can be written in the form $P_{\mathbb{W}_4}(A) = P_1(A)\, i_1 + P_2(A)\, i_2 + P_3(A)\, i_3 + P_4(A)\, i_4$, where $P_1(A), P_2(A), P_3(A), P_4(A)$ are probabilities.

The topology induced by the bihyperbolic norm generates the Borel σ-algebra $\mathfrak{B}_{\mathbb{W}_4}$ in $\mathbb{W}_4$.

Definition 4. Let $(\Omega, \Sigma, P_{\mathbb{W}_4})$ be a $\mathbb{W}_4$-probability space. A function $X_{\mathbb{W}_4}(\omega) : \Omega \to \mathbb{W}_4$ such as $X_{\mathbb{W}_4}^{-1}(A) \in \Sigma$ for each open set A in $\mathbb{W}_4$ is called a $\mathbb{W}_4$-valued random variable.

Each $\mathbb{W}_4$-valued random variable $X_{\mathbb{W}_4}(\omega)$ can be written in the following form $X_{\mathbb{W}_4}(\omega) = X_1(\omega)\, i_1 + X_2(\omega)\, i_2 + X_3(\omega)\, i_3 + X_4(\omega)\, i_4$, where $X_1(\omega), X_2(\omega), X_3(\omega), X_4(\omega)$ are $\mathbb{R}$-random variables on Ω.

Theorem 5. *The $\mathbb{W}_4$-valued function $X_{\mathbb{W}_4}(\omega)$ on a measurable space (Ω, Σ) is a $\mathbb{W}_4$-valued random variable if and only if $\{\omega \in \Omega | X_{\mathbb{W}_4}(\omega) \prec x \text{ or } X_{\mathbb{W}_4}(\omega) \in A_x\} \in \Sigma$ for all $x \in \mathbb{W}_4$.*

Theorem 6. *Let $X_{\mathbb{W}_4}(\omega)$ be a $\mathbb{W}_4$-valued random variable on $(\Omega, \Sigma, P_{\mathbb{W}_4})$. For $\forall x \in \mathbb{W}_4$ the following conditions are equivalent:* $\{\omega \in \Omega | X_{\mathbb{W}_4}(\omega) \preccurlyeq x\} \in \Sigma$; $\{\omega \in \Omega | X_{\mathbb{W}_4}(\omega) \succ x$ *or* $X_{\mathbb{W}_4}(\omega) \in A_x\} \in \Sigma$; $\{\omega \in \Omega | X_{\mathbb{W}_4}(\omega) \succcurlyeq x$ *or* $X_{\mathbb{W}_4}(\omega) \in A_x\} \in \Sigma$; $\{\omega \in \Omega | X_{\mathbb{W}_4}(\omega) \prec x\} \in \Sigma$.

The algebra of quaternions is a structure of the form $\mathbb{H} = \{a_0 + a_1 i + a_2 j + a_3 k,\ a_i \in \mathbb{R},\ i = 0, 1, 2, 3\}$, where $i^2 = j^2 = k^2 = -1,\ ij = -ji = k,\ jk = -kj = i,\ ki = -ik = j$.

Definition 7. Let $\mathfrak{M}$ be a σ-algebra of subsets of a non-empty set X. A quaternionic measure ω on a measurable space $(X, \mathfrak{M})$ is a quaternion-valued function on $\mathfrak{M}$ such that for any collection of sets $\{A_n, n \in \mathbb{N}\} \subset \mathfrak{M}$ that $A_n \cap A_m = \emptyset$ whenever $n \neq m$ we have $\omega\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \omega\left(A_n\right)$.

Definition 8. The function of the sets $\operatorname{var}[\omega](A) := \sup \sum_{n=1}^{\infty} |\omega(A_n)|$ is defined on the $\mathfrak{M}$, where the supremum is taken for all partitions of A, we call the complete variation ω.

It is clear that $|\omega(A)| \leq var[\omega](A)$.

Theorem 9. *The total variation* $\operatorname{var}[\omega]$ *of a quaternionic measure ω on a measurable space $(X, \mathfrak{M})$ is a positive measure on $(X, \mathfrak{M})$.*

Theorem 10. *If ω is a quaternionic measure on a measurable space $(X, \mathfrak{M})$, then* $\operatorname{var}[\omega](X) < \infty$.

Definition 11. Let μ be a positive measure and ω be a quaternionic measure on a measurable space $(X, \mathfrak{M})$. We say that ω is absolutely continuous with respect to μ if $\mu(A) = 0$ implies $\omega(A) = 0$ for $A \in \mathfrak{M}$. We write $\omega \ll \mu$.

Definition 12. Given a quaternionic measure ω on a measurable space $(X, \mathfrak{M})$, assume that there is a set $F \in \mathfrak{M}$ such that $\omega(A) = \omega(A \cap F)$ for every $A \in \mathfrak{M}$, we say that ω is concentrated on F. This is equivalent to say that $\omega(A) = 0$ whenever $A \cap F = 0$.

Let ω_1, ω_2 be quaternionic measures on $\mathfrak{M}$ and suppose there exist a pair of disjoint sets F, G such that ω_1 is concentrated on F and ω_2 is concentrated on G. Then we say that ω_1 and ω_2 are mutually singular, and write $\omega_1 \perp \omega_2$.

Theorem 13. *Let λ be a signed real σ-finite measure on a measurable space $(X, \mathfrak{M})$ and let ω be a quaternionic measure on $(X, \mathfrak{M})$. Then there exists a unique pair of quaternionic measures ω_a and ω_s such that $\omega = \omega_a + \omega_s$, $\omega_a \ll \lambda$, $\omega_s \perp \lambda$. The pair ω_a, ω_s is called the Lebesgue decomposition of ω w.r.t. λ, where ω_a is the absolutely continuous part and ω_s is the singular part of the decomposition.*

Theorem 14. *(Radon-Nikodym). Let μ be a positive σ-finite measure on a measurable space $(X, \mathfrak{M})$, let ω be a quaternionic measure on $(X, \mathfrak{M})$ and let ω_a be absolutely continuous part of the Lebesgue decomposition of ω w.r.t. μ. Then there is a measurable quaternionic function $h(x)$ on X such that for every set $A \in \mathfrak{M}$ $\omega_a(A) = \int_A h(x)\, d\mu$, where $h(x)$ is uniquely defined up to a μ-null set.*

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The invariants of planar 3-webs with respect to group of symplectic diffeomorphisms, for the case of the conformal group

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The classical web geometry ([1],[2],[4]) studies invariants of foliation families with respect to pseudogroup of diffeomorphisms. Thus for the case of planar 3-webs the basic semi invariant is the Blaschke curvature ([3]). It is also curvature of the Chern connection ([4]) that are naturally associated with a planar 3-web. Remark that we have in addition to the diffeomorphism group two infinite dimensional groups: symplectic and conformal groups.

We investigate invariants of planar 3-webs with respect to group of symplectic diffeomorphisms, for the case of the conformal group see ([5]). We found the basic symplectic invariants of planar 3-webs that allow us to solve the symplectic equivalence problem for planar 3-webs in general position. The Lie-Tresse theorem ([2]) gives the complete description of the field of rational symplectic differential invariants of planar 3-webs. We also give normal forms for homogeneous 3-webs, i.e. 3-webs having constant basic invariants.

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Topology of spaces of smooth functions and gradient-like flows with prescribed singularities on surfaces

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Let M be a smooth orientable connected closed two-dimensional surface, and $f_0 \in \mathcal{F}(M)$ a function all whose critical points have types A_i, D_j, E_k. Consider the set $\mathcal{F} = \mathcal{F}(f_0)$ of all functions $f \in C^\infty(M)$ having the same types of local singularities as f_0. Denote by $\mathcal{D}^0(M)$ the component of unity in the group $\mathcal{D}(M) = \mathrm{Diff}^+(M)$ of orientation-preserving diffeomorphisms. The group $\mathcal{D}(\mathbb{R}) \times \mathcal{D}(M)$ acts on the space $\mathcal{F}$ by "left-right changes of coordinates".

We want to describe the topology of the space $\mathcal{F}$, equipped with the C^∞-topology, and its decomposition into $\mathcal{D}^0(\mathbb{R}) \times \mathcal{D}^0(M)$- and $\mathcal{D}^0(M)$-orbits. This problem was solved by the author in the cases when either f_0 is a Morse function and $\chi(M) < 0$ [2, 3], or all critical points of f_0 have A_μ types, $\mu \in \mathbb{N}$ [4]. Topology of the $\mathcal{D}^0(M)$-orbits was studied by S.I. Maksymenko [5] (allowing some other types of degenerate singularities) and by the author [2, 3, 4] (for A_μ-singularities).

For any function $f \in \mathcal{F}$, consider the set $\mathcal{C}_f := \{P \in M \mid \mathrm{d}f(P) = 0\}$ of its critical points. These critical points form five classes of topological equivalence (some classes may be empty):

$$\mathcal{C}_f^{min} = \bigcup_{i\geq 1} A_{2i-1}^{+,+}(f),\ \mathcal{C}_f^{max} = \bigcup_{i\geq 1} A_{2i-1}^{+,-}(f),\ \mathcal{C}_f^{saddle} = A_1^-(f) \cup \bigcup_{\eta=\pm} (\bigcup_{i\geq 2} A_{2i-1}^{-,\eta}(f) \cup D_{2i+1}^\eta(f)) \cup E_7^\eta(f),$$

$$\mathcal{C}_f^{triv} = (\bigcup_{i\geq 1,\eta=\pm} A_{2i}^\eta(f)) \cup (\bigcup_{j\geq 2} D_{2j}^+(f)) \cup E_6^+(f) \cup E_6^-(f) \cup E_8^+(f) \cup E_8^-(f), \quad \mathcal{C}_f^{mult} = \bigcup_{j\geq 2} D_{2j}^-(f),$$

i.e. the critial points of local minima, local maxima, saddle points, quasi- and multy-saddle points, respectively. Here $A_i^{\pm,\pm}(f)$, $D_j^\pm(f)$ and $E_k^\pm(f)$ denote the corresponding subsets of critical points of $A - D - E$ types. In the set $\mathcal{C}_f^{extr} := \mathcal{C}_f^{min} \cup \mathcal{C}_f^{max}$ of local extremum points, consider the subset $\mathcal{C}_f^{extr*}$ of degenerate (non-Morse) critical points.

Denote $s := \max\{0, \chi(M) + 1\}$.

Theorem 1. *For any function $f_0 \in C^\infty(M)$, whose all critical points have $A - D - E$ types, the space $\mathcal{F} = \mathcal{F}(f_0)$ has the homotopy type of a manifold $\mathbb{B} = \mathbb{B}(f_0)$ having dimension $\dim \mathbb{B} = 2s + |\mathcal{C}_{f_0}^{extr}| + |\mathcal{C}_{f_0}^{extr*}| + 2|\mathcal{C}_{f_0}^{triv}| + 3|\mathcal{C}_{f_0}^{saddle}| + 4|\mathcal{C}_{f_0}^{mult}|$. Moreover:*

(a) *There exists a surjective submersion $\kappa : \mathcal{F} \to \mathbb{B}$ and a stratification (respectively, a fibration of codimension $|\mathcal{C}_{f_0}|$) on $\mathbb{B}$ such that every $\mathcal{D}^0(\mathbb{R}) \times \mathcal{D}^0(M)$-orbit (resp., $\mathcal{D}^0(M)$-orbit) in $\mathcal{F}$ is the κ-preimage of a stratum (resp., a fiber) in $\mathbb{B}$.*

(b) *The map κ provides a homotopy equivalence between any $\mathcal{D}^0(M)$-invariant subset $I \subseteq \mathcal{F}$ and its image $\kappa(I) \subseteq \mathbb{B}$. In particular, it provides homotopy equivalences between $\mathcal{F}$ and $\mathbb{B}$, and between every $\mathcal{D}^0(\mathbb{R}) \times \mathcal{D}^0(M)$-orbit (resp., $\mathcal{D}^0(M)$-orbit) from $\mathcal{F}$ and the corresponding stratum (resp., fiber) in $\mathbb{B}$.*

In particular, $\pi_k(\mathcal{F}) \cong \pi_k(\mathbb{B})$, $H_k(\mathcal{F}) \cong H_k(\mathbb{B})$. Thus $H_k(\mathcal{F}) = 0$ for all $k > \dim \mathbb{B}$.

Our proof of Theorem 1 uses a result (obtained in collaboration with Alexandra Orevkova) about a "uniform" reduction of a smooth function to a normal form near its critical points.

Suppose $\Omega \in \Lambda^n(M)$ is a volume form on a n-manifold $M = M^n$. Let $\mathcal{P} \subset M$ be a finite subset. For any vector field ξ on $M' := M \setminus \mathcal{P}$, we assign the $(n-1)$-form $\beta = i_\xi \Omega \in \Lambda^{n-1}(M')$. Clearly, this assignement is one-to-one. Furthermore, the flow of the vector field ξ is volume-preserving if and only β is a closed form. Indeed: the Lie derivative $L_\xi \Omega = (i_\xi \mathrm{d} + \mathrm{d} i_\xi)\Omega = \mathrm{d} i_\xi \Omega = \mathrm{d}\beta$, so the

Lie derivative vanishes if and only if $\mathrm{d}\beta = 0$. By abusing language, we will call the $(n-1)$-form β a *flow*.

Suppose now that $n = \dim M = 2$. A closed 1-form β on $M' = M \setminus \mathcal{P}$ will be called a *gradient-like flow* on M if there exists a Morse function $f \in C^\infty(M)$, called an *energy function* of β, such that

(i) the set $\mathcal{P}$ coincides with the set of local extremum points of f,

(ii) the 3-form $\mathrm{d}f \wedge \beta|_{M \setminus \mathcal{C}_f}$ has no zeros and defines a positive orientation on M,

(iii) in a neighbourhood of every point $P \in \mathcal{C}_f$, there exist local coordinates x, y such that either $\beta = \mathrm{d}(xy)$, $f = f(P) + x^2 - y^2$ and $P \in \mathcal{Z} = \mathcal{C}_f \setminus \mathcal{P}$, or $\beta = (x\mathrm{d}y - y\mathrm{d}x)/(x^2+y^2)$, $f = f(P) \pm (x^2+y^2)$ and $P \in \mathcal{P}$.

Geometrically, the set $\mathcal{P}_\beta := \mathcal{P}$ consists of *sourses* and *sinks* of the flow β and coincides with the set of local extremum points of the energy function f, while the set $\mathcal{Z}_\beta := \mathcal{Z} = \mathcal{C}_f \setminus \mathcal{P}$ consists of *saddle points* of the flow β and coincides with the set of saddle critical points of f.

Denote by $\mathcal{B}(M)$ the space of all gradient-like flows β on M (having arbitrary finite sets $\mathcal{Z} = \mathcal{Z}_\beta$ and $\mathcal{P} = \mathcal{P}_\beta$ of saddles, sourses and sinks depending on β). Endow this space with C^∞-topology. For a gradient-like flow $\beta_0 \in \mathcal{B}(M)$, denote by $\mathcal{B}(\beta_0)$ the set of all gradient-like flows $\beta \in \mathcal{B}(M)$ having the same local singularities as β_0 (in particular, $|\mathcal{Z}_\beta| = |\mathcal{Z}_{\beta_0}|$ and $|\mathcal{P}_\beta| = |\mathcal{P}_{\beta_0}|$).

We want to describe the topology of the space $\mathcal{B}(\beta_0)$, equipped with the C^∞-topology, and its decomposition into $\mathcal{D}^0(M)$-orbits and into classes of (orbital) topological equivalence.

Theorem 2. *For any gradient-like flow β_0 on M, the space $\mathcal{B}(\beta_0)$ has the homotopy type of the manifold $\mathbb{B} = \mathbb{B}(f_0)$ from Theorem 3, where f_0 is an energy function of β_0. Moreover:*

(a) *There exists a surjective submersion $\lambda : \mathcal{B}(\beta_0) \to \mathbb{B}$, a stratification and a $(|\mathcal{P}_{\beta_0}| + |\mathcal{Z}_{\beta_0}|)$-dimensional fibration on $\mathbb{B}$ such that every class of topological equivalence (resp., every $\mathcal{D}^0(M)$-orbit) in $\mathcal{B}(\beta_0)$ is the λ-preimage of the stratum (resp., the fibre) from $\mathbb{B}$.*

(b) *The map λ provides a homotopy equivalence between every $\mathcal{D}^0(M)$-invariant subset $I \subseteq \mathcal{B}(\beta_0)$ and its image $\lambda(I) \subseteq \mathbb{B}$. In particular, it provides a homotopy equivalence between $\mathcal{B}(\beta_0)$ and $\mathbb{B}$, as well as between every class of topological equivalence (resp., every $\mathcal{D}^0(M)$–orbit) in $\mathcal{B}(\beta_0)$ and the corresponding stratum (resp., fibre) in $\mathbb{B}$. All fibres and strata in $\mathbb{B}$ (and, thus, all classes of topological equivalence and all $\mathcal{D}^0(M)$–orbits in $\mathcal{B}(\beta_0)$) are homotopy equivalent either to a point, or to T^2, or to $SO(3)/G$ or to S^2, in dependence on whether $\chi(M) < 0$, or $\chi(M) = 0$, or $\chi(M) \cdot |\mathcal{Z}_{\beta_0}| > 0$, or $|\mathcal{Z}_{\beta_0}| = 0$, respectively, where G is a finite subgroup of $SO(3)$.*

In particuar, $\pi_k(\mathcal{B}(\beta_0)) \cong \pi_k(\mathbb{B})$, $H_k(\mathcal{B}(\beta_0)) \cong H_k(\mathbb{B})$. Thus $H_k(\mathcal{B}(\beta_0)) = 0$ for all $k > \dim \mathbb{B}$.

We will illustrate our results on several examples.

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Nonlocal problem with integral conditions for homogeneous system of partial differential equations of second order

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Let $\Pi(T) = \{(t,x) \in \mathbb{R}^2 : t \in [0,T], x \in \mathbb{R}\}$, $T > 0$. Let us denote $E_{\alpha,\beta}, \alpha > 0, \beta > 0.$, to the space of functions $\varphi \in L_2(\mathbb{R})$, with the finite norm [1]

$$\|\varphi\|_{E_{\alpha,\beta}} = \sqrt{\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} |\hat{\varphi}(\xi)|^2 \left(1+|\xi|\right)^{2\alpha} \exp(2\beta|\xi|)d\xi}$$

where $\hat{\varphi}(\xi)$ is the Fourier transform of the function $\varphi(x)$. In the strip $\Pi(T)$ we consider nonlocal-integral problem

$$L\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right) u(t,x) \equiv \frac{\partial^n u(t.x)}{\partial t^n} + \sum_{j=1}^{n} a_j \frac{\partial^n u(t,x)}{\partial t^{n-j}\partial x^j} = 0, \quad a_j \in \mathbb{C}, \quad (t,x) \in \Pi(T), \tag{1}$$

$$\left.\frac{\partial^k U}{\partial t}\right|_{t=0} - \left.\frac{\partial^k U}{\partial t}\right|_{t=T} + \int\limits_0^T t^k U(t,x)dt = \varphi_k(x), \quad k = 0,...,n-2, \tag{2}$$

$$\int\limits_0^{T_1} t^{n-1} U(t,x)dt + \int\limits_{T_2}^{T} t^{n-1} U(t,x)dt = \varphi_{n-1}(x) \tag{3}$$

where $a_1, a_2 \in \mathbb{C}$. Assuming that the real parts of roots of polynomial $\lambda^n + a_1\lambda^{n-1} + ... + a_n$ are nonzero and different, the correctioness of the problem (1) - (3) in the space of functions $C^2([0,T], E_{\alpha,\beta})$ is established.

Obtained results continue the research of the work [2].

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Realization of groups as fundamental groups of orbits of smooth maps

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Let M be a connected compact oriented surface and P be a real line $\mathbb{R}$ or a circle S^1. Note, that the group $\mathcal{D}(M)$ of diffeomorphisms of M naturally acts on the space of smooth maps $C^\infty(M,P)$ by the rule $(f,h) \longmapsto f \circ h$, where $h \in \mathcal{D}(M)$, $f \in C^\infty(M,P)$. For $f \in C^\infty(M,P)$ denote by $\mathcal{O}(f)$ the orbit of f under this action. Let $\mathcal{M}(M,P)$ be the set of isomorphism clasess of fundamental groups $\pi_1\mathcal{O}(f)$ of orbits of Morse maps $f\colon M \to P$.

S. Maksymenko [1, 2] and B. Feshchenko [3] introduced the sets of isomorphism classes $\mathcal{B}$ and $\mathcal{T}$ of groups generated by direct products and certain wreath products. They have proved that $\mathcal{M}(M,P) \subset \mathcal{B}$ if M is different from a 2-sphere S^2 and a 2-torus T^2, and $\mathcal{M}(T^2,\mathbb{R}) \subset \mathcal{T}$. We proved that these inclusions are equalities.

Definition 1. Let $\mathcal{B}$ be a minimal class of groups satisfying the following conditions:

1) $1 \in \mathcal{B}$;
2) if $G_1, G_2 \in \mathcal{B}$, then $G_1 \times G_2 \in \mathcal{B}$;
3) if $G \in \mathcal{B}$ i $n \geq 1$, then $G \wr_n \mathbb{Z} \in \mathcal{B}$.

Let also $\mathcal{T}$ be the set of isomorphism classes of groups consisting of groups of the form $G \wr_{n,m} \mathbb{Z}^2$, where $G \in \mathcal{B}$ and $n, m \geq 1$.

Let also $\mathcal{B}^O$ be a subclass of $\mathcal{B}$ consisting of groups $(A \times B) \wr_n \mathbb{Z}$, where $A, B \in \mathcal{B} \setminus \{1\}$ and $n \geq 1$. Note, that $\mathcal{B}^O \subset \mathcal{B} \subset \mathcal{T}$.

Denote by $\mathcal{F}(M,P)$ the space of smooth maps $f \in C^\infty(M,P)$ satisfying the following two conditions:
(1) all critical points of f belong to the interior of M, and f takes constant values on each connected component of the boundary of M;
(2) for each critical point z of f its germ at z is smoothly equivalent to some non-zero homogeneous polynomial $\mathbb{R}^2 \to \mathbb{R}$ of degree ≥ 2 without multiple factors.

The set of all Morse maps from M to P is denoted by $Morse(M,P)$. For each map $f \in \mathcal{F}(M,P)$ we can define the (continuous) function ε_f from the set of connected components of the boundary ∂M to $\{\pm 1\}$, which takes the value -1 on the boundary component if f has a local minimum on this component, and $+1$ if f has a local maximum on this component. Let $\mathcal{E}_M$ be the set of all continuous functions $\varepsilon\colon \partial M \to \{\pm 1\}$. For $\varepsilon \in \mathcal{E}_M$ we denote by $\mathcal{F}(M,P,\varepsilon)$ ($Morse(M,P,\varepsilon)$) subset of $\mathcal{F}(M,P)$ ($Morse(M,P)$) of functions f, for which $\varepsilon_f = \varepsilon$.

Denote

$\mathcal{G}_X(M,P,\varepsilon) := \{\pi_1\mathcal{O}(f,X) \mid f \in \mathcal{F}(M,P,\varepsilon)\}$,

$\mathcal{M}_X(M,P,\varepsilon) := \{\pi_1\mathcal{O}(f,X) \mid f \in Morse(M,P,\varepsilon)\}$,

$\mathcal{G}^\Psi := \{\pi_1\mathcal{O}(f) \mid f \in \mathcal{F}(T^2,\mathbb{R}), \text{the Kronrod-Reeb graph } \Gamma_f \text{ is a tree}\}$,

$\mathcal{M}^\Psi := \{\pi_1\mathcal{O}(f) \mid f \in Morse(T^2,\mathbb{R}), \text{the Kronrod-Reeb graph } \Gamma_f \text{ is a tree}\}$,

$\mathcal{G}^O := \{\pi_1\mathcal{O}(f) \mid f \in \mathcal{F}(T^2,\mathbb{R}), \text{the Kronrod-Reeb graph } \Gamma_f \text{ has an unique cycle}\}$,

$\mathcal{M}^O := \{\pi_1\mathcal{O}(f) \mid f \in Morse(T^2,\mathbb{R}), \text{the Kronrod-Reeb graph } \Gamma_f \text{ has an unique cycle}\}$.

Theorem 2. (1) *Let M be a connected compact oriented surface distinct from 2-torus and 2-sphere, and let $\varepsilon\colon \partial M \to \{\pm 1\}$ be an arbitrary map from $\mathcal{E}_M$. Then*

a) *if $M = S^1 \times [0,1]$, and ε is constant, i.e takes the same value on components of the boundary ∂M, then $\mathcal{M}_{\partial M}(M,P,\varepsilon) = \mathcal{G}_{\partial M}(M,P,\varepsilon) = \mathcal{B} \setminus \{1\}$,*

b) *if $M = S^1 \times [0,1]$ and ε takes different values on the components of the boundary ∂M or $M \neq S^1 \times [0,1]$, then $\mathcal{M}_{\partial M}(M,P,\varepsilon) = \mathcal{G}_{\partial M}(M,P,\varepsilon) = \mathcal{B}$.*

(2) *There are equalities $\mathcal{M}^\Psi = \mathcal{G}^\Psi = \mathcal{T}$, $\mathcal{M}^O = \mathcal{G}^O = \mathcal{B}^O$.*

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Modified quaternionic operator calculus and its application to micropolar elasticity

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Original ideas for the extension of classical elasticity theory to account microeffects of a continuum go back to the work [1] of Cosserat brothers, where they introduced a new theory called the *Cosserat continuum*. The introduced theory grabbed attention of many scientists. Among others, works of Eringen [2], and Nowacki [3] significantly supported further development of the theory. Eringen introduced micro-inertia in the theory, which has led to renaming of the theory to the *micropolar elasticity*. From practical point of view, the micropolar theory models not only displacements of a continuum, as in the classical theory of elasticity, but also its rotations.

In this talk we introduce representation formulae for the solution of spatial boundary value problems of micropolar elasticity. The representation formulae are constructed in the framework of quaternionic analysis, which is a natural extension of the classical complex analysis to higher dimensions. The main toolbox for constructing representation formulae for boundary value problems of mathematical physics in hypercomplex analysis is the co-called quaternionic operator calculus [4, 5]. The essential ingredient is the T-operator (Teodorescu transform), which is a right inverse to the generalised Cauchy-Riemann operator. Accomplishing the T-operator with the F-operator (Cauchy-Bitsadze operator), the higher-dimensional generalisation of the classical Borel-Pompeiu formula can be obtained.

In this talk, we consider the following boundary value problem:

Problem 1. Let $\Omega \subset \mathbb{R}^3$ be a bounded simply connected domain with a sufficiently smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$. A boundary value problem of the micropolar elasticity is formulated as follows

$$(\lambda + 2\mu + \kappa)\,\nabla\nabla\cdot\mathbf{u} - (\mu+\kappa)\,\nabla\times\nabla\times\mathbf{u} = -\kappa\nabla\times\boldsymbol{\varphi}, \tag{1}$$

$$(\alpha+\beta+\gamma)\,\nabla\nabla\cdot\boldsymbol{\varphi} - \gamma\nabla\times\nabla\times\boldsymbol{\varphi} - 2\kappa\boldsymbol{\varphi} = -\kappa\nabla\times\mathbf{u}, \tag{2}$$

with boundary conditions

$$\begin{cases} \mathbf{u} &= \mathbf{g}_1 & \text{on } \Gamma_0, \\ \boldsymbol{\varphi} &= \mathbf{g}_2 & \text{on } \Gamma_0, \end{cases} \quad \text{and} \quad \begin{cases} t_{lk}n_l &= t_{(\mathbf{n})k} & \text{on } \Gamma_1, \\ m_{lk}n_l &= m_{(\mathbf{n})k} & \text{on } \Gamma_1, \end{cases} \tag{3}$$

where $\mathbf{u}$ is the displacement vector, $\boldsymbol{\varphi}$ is the vector of micropolar rotation, t_{lk} is the stress tensor, m_{lk} is the couple stress tensor, ρ is the material density, j is a rotational inertia, λ and μ are the Lamé parameters, $\kappa, \alpha, \beta, \gamma$ are material parameters of micropolar theory, n_j are components of the unit outer normal vector, $t_{(\mathbf{n})k}$ are given surface forces, and $m_{(\mathbf{n})k}$ are given surface moments.

After that, a quaternionic formulation of the boundary value problem (1)-(3) is considered [6]:

Proposition 2. *Considering the displacement field* $\mathbf{u} \in C^2(\Omega)$ *and micropolar rotations* $\boldsymbol{\varphi} \in C^2(\Omega)$ *as pure quaternions, i.e.* $\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3$, $\boldsymbol{\varphi} = \varphi_1\mathbf{e}_1 + \varphi_2\mathbf{e}_2 + \varphi_3\mathbf{e}_3$, *equations of micropolar elasticity (1)-(2) can be written as follows*

$$\begin{aligned} D\,M_1\,D\,\mathbf{u} + \kappa\mathrm{Vec}\,D\boldsymbol{\varphi} &= 0, \\ \left(D - i\sqrt{\tfrac{2\kappa}{\gamma}}\right) M_2 \left(D + i\sqrt{\tfrac{2\kappa}{\gamma}}\right)\boldsymbol{\varphi} + \kappa\mathrm{Vec}\,D\mathbf{u} &= 0, \end{aligned} \tag{4}$$

where the operators M_1 and M_2 are defined by

$$\begin{aligned} M_1\mathbf{w} &:= -(\lambda+2\mu+\kappa)w_0-(\mu+\kappa)w_1\mathbf{e}_1-(\mu+\kappa)w_2\mathbf{e}_2 \\ &\quad -(\mu+\kappa)w_3\mathbf{e}_3, \\ M_2\mathbf{w} &:= -(\alpha+\beta+\gamma)w_0-\gamma\, w_1\mathbf{e}_1-\gamma\, w_2\mathbf{e}_2-\gamma\, w_3\mathbf{e}_3, \end{aligned}$$

for a quaternion-valued function $\mathbf{w}=w_0+w_1\mathbf{e}_1+w_2\mathbf{e}_2+w_3\mathbf{e}_3$.

By reformulating the system as a system of operator equations, the questions of existence, regularity, stablity and uniqueness can be studied by using the classical and modified versions of quaternionic operator calculus [4, 5]:

Theorem 3. *The system of equations*

$$\left\{ \begin{aligned} D\,M_1\,D\,\mathbf{u}+\kappa\mathrm{Vec}\,D\boldsymbol{\varphi} &= 0, \\ \left(D-i\sqrt{\tfrac{2\kappa}{\gamma}}\right)M_2\left(D+i\sqrt{\tfrac{2\kappa}{\gamma}}\right)\boldsymbol{\varphi}+\kappa\mathrm{Vec}\,D\mathbf{u} &= 0, \end{aligned} \right. \tag{5}$$

with Dirichlet boundary conditions

$$\left\{ \begin{aligned} \mathbf{u} &= \mathbf{g}_1 \quad \text{on } \Gamma_0, \\ \boldsymbol{\varphi} &= \mathbf{g}_2 \quad \text{on } \Gamma_0, \end{aligned} \right.$$

is equivalent to the system of operator equations

$$\left\{ \begin{aligned} \mathbf{u} &= F_\Gamma\tilde{\mathbf{g}}_1+TM_1^{-1}F_\Gamma(\mathrm{tr}\,TM_1^{-1}F_\Gamma)^{-1}Q_\Gamma\tilde{\mathbf{g}}_1 \\ &\quad -\kappa\,TM_1^{-1}T\,\mathrm{Vec}\,D\boldsymbol{\varphi}, \\ \\ \boldsymbol{\varphi} &= F_\alpha\tilde{\mathbf{g}}_2+T_\alpha M_2^{-1}F_{-\alpha}\left(\mathrm{tr}\,T_\alpha M_2^{-1}F_{-\alpha}\right)^{-1}Q_\alpha\tilde{\mathbf{g}}_2 \\ &\quad -\kappa\,T_\alpha M_2^{-1}T_{-\alpha}\mathrm{Vec}\,D\mathbf{u}, \end{aligned} \right. \tag{6}$$

where $\tilde{\mathbf{g}}_1=\mathbf{g}_1+\kappa\,\mathrm{tr}\,TM_1^{-1}T\,\mathrm{Vec}\,D\varphi$ *and* $\tilde{\mathbf{g}}_2=\mathbf{g}_1+\kappa\,\mathrm{tr}\,T_\alpha M_2^{-1}T_{-\alpha}\mathrm{Vec}\,D\mathbf{u}$.

Further results related to the uniqueness of solution, as well as the estimate for a difference between the micropolar model and the classical model of elasticity, will be presented in the talk.

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On non-Hausdorff manifolds of dimension 1

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Consider a T_1 topological space Y that is locally homeomorphic with $[0,1)$. In other words, Y is a one-dimensional non-Hausdorff but T_1 manifold.

As usual, a point $y \in Y$ is *internal*, if it has an open neighborhood homeomorphic with $(0,1)$. Otherwise, y has an open neighborhood homeomorphic with $[0,1)$ and is called a *boundary* point. The set of all internal and boundary points will be denoted by $\mathrm{Int}Y$ and ∂Y respectively.

For a point $z \in Y$ define its *Hausdorff closure*, $\mathrm{hcl}(z)$, to be the intersection of closures of all neighbourhoods of z, that is

$$\mathrm{hcl}(z) := \bigcap_{V \text{ is a neighbourhood of } z} \overline{V}.$$

Evidently, $z \in \mathrm{hcl}(z)$. We say that z is *special* whenever $\mathrm{hcl}(z) \setminus z \neq \varnothing$. Denote by $\mathcal{V}$ the set of all special points of Y.

Let $\mathcal{H}(Y)$ be the group of homeomorphisms of Y endowed with compact open topology, and $\mathcal{H}_{\mathrm{id}}(Y)$ be the identity path component of $\mathcal{H}(Y)$, so it is a normal subgroup consisting of homeomorphisms isotopic to the identity. The following statement gives a characterization of $\mathcal{H}_{\mathrm{id}}(Y)$ under assumption that the set $\mathcal{V}$ of special points of Y is locally finite.

Theorem 1. *Let Y be a second-countable T_1 topological space being locally homeomorphic with $[0,1)$ and such that the set $\mathcal{V}$ of its special points is locally finite. Let also $k \in \mathcal{H}(Y)$. Then $k \in \mathcal{H}_{\mathrm{id}}(Y)$ if and only if the following two conditions hold:*

(1) *k fixes each special point of Y;*
(2) *k preserves orientation of each connected component e of $Y \setminus \mathcal{V}$.*

Symplectomorphisms preserving smooth functions on surfaces

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Let M be a compact connected surface and P is a connected one-dimensional manifold without boundary, i.e. either the real line $\mathbb{R}$ or the circle S^1. Denote by $\mathcal{D}(M)$ the group of all smooth (C^∞) diffeomorphisms of M. This group acts from the right on the space $C^\infty(M,P)$ by the following rule: if $h \in \mathcal{D}(M)$ and $f \in C^\infty(M,P)$, then the result of the action of h on f is the composition map $f \circ h : M \to P$. For $f \in C^\infty(M,P)$ let Σ_f be the set of its critical points, and

$$\mathcal{S}(f) = \{h \in \mathcal{D}(M) \mid f \circ h = f\},$$
$$\mathcal{O}(f) = \{f \circ h \mid h \in \mathcal{D}(M)\}$$

be respectively the *stabilizer* and the *orbit* of f under that action. Endow these spaces with C^∞ topologies and denote by $\mathcal{D}_{\mathrm{id}}(M)$ and $\mathcal{S}_{\mathrm{id}}(f)$ the corresponding path components of id_M in $\mathcal{D}(M)$ and $\mathcal{S}(f)$, and by $\mathcal{O}_f(f)$ the path component of $\mathcal{O}(f)$ containing f. We will omit X from notation whenever it is empty.

Notice that $\mathcal{S}_{\mathrm{id}}(f)$ is a normal subgroup of $\mathcal{S}(f)$, and the quotient:

$$\pi_0\mathcal{S}(f) := \mathcal{S}(f)/\mathcal{S}_{\mathrm{id}}(f)$$

is the group of path components of $\mathcal{S}(f)$. This group is an analogue of mapping class group for f-preserving diffeomorphisms.

Let $\mathcal{F}(M,P)$ be a subset of $C^\infty(M,P)$ consisting of maps satisfying the following two axioms:

(B) The map f takes a constant value at each connected component of ∂M and has no critical points in ∂M;

(L) For every critical point z of f, there are local coordinates in which f is a homogeneous polynomial $\mathbb{R}^2 \to \mathbb{R}$ of degree ≥ 2 without multiple factors.

Evidently, $\mathcal{F}(M,P)$ contains all Morse maps.

For $f \in \mathcal{F}(M,P)$ the homotopy types of $\mathcal{S}_{\mathrm{id}}(f)$ and orbits were described by S. Maksymenko, and the homotopy types of connected components of orbit $\mathcal{O}(f)$ by S. Maksymenko, E. Kudryavtseva (for Morse maps and for smooth functions $f : M \to \mathbb{R}$ with *simple singularities* which are not homogeneous but quasi-homogeneous), B. Feshchenko, I. Kuznietsova, Yu. Soroka, A. Kravchenko.

Theorem 1. *Let $f \in \mathcal{F}(M,P)$. Then the natural map $p : \mathcal{S}(f) \to \pi_0\mathcal{S}(f)$ has a section:*

$$s : \pi_0\mathcal{S}(f) \to \mathcal{S}(f),$$

so s is a homomorphism such that $p \circ s = \mathrm{id}_{\pi_0\mathcal{S}_{\mathrm{id}}(f)}$.

Moreover, if M is orientable, then there exists a symplectic structure, i.e. a non-degenerate differential 2-form ω, on M, such that the image $s\big(\pi_0\mathcal{S}(f)\big)$ consists of symplectic diffeomorphisms with respect to ω.

Second classical Zariski topology of multiplicational module

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Let R be a associative ring and M an multiplicative R-module. If N is a subset of an R-module M we write $N \leq M$ to indicate that N is a submodule of M.

Definition 1. Proper submodule P of the left module M is called **prime submodule**, if quotient module M/P is prime left module, ie $Ann(K/P) = Ann(M/P)$ for every nonzero submodule K/P of module M/P.

This definition can be found in such papers: [1], [2], and there are a lot of interesting results about such modules. Set of all prime submodules of module M is called prime spectrum of module M and is denoted by $Spec(M)$.

Definition 2. A non-zero submodule N of M is said to be second if for each $a \in R$, the homomorphism $N \to^a N$ is either surjective or zero [3]. More information about this class of modules can be found in [4].

Let $Spec^s(M)$ be the set of all second submodules of M. For any submodule N of M, $V^{s*}(N)$ is defined to be the set of all second submodules of M contained in N. Of course, $V^{s*}(0)$ is just the empty set and $V^{s*}(M)$ is $Spec^s(M)$. It is easy to see that for any family of submodules $N_i (i \in I)$ of M, $\cap_{i\in I} V^{s*}(N_i) = V^{s*}(\cap_{i\in I} N_i)$. Thus if $\zeta s*(M)$ denotes the collection of all subsets $V^{s*}(N)$ of $Spec^s(M)$, then $\zeta s*(M)$ contains the empty set and $Spec^s(M)$, and $\zeta s*(M)$ is closed under arbitrary intersections. In general $\zeta s*(M)$ is not closed under finite unions.
Now let N be a submodule of M. We define $W^s(N) = Spec^s(M) - V^{s*}(N)$ and put $\Omega^s(M) = \{W^s(N) : N \leq M\}$. Let $\eta^s(M)$ be the topology on $Spec^s(M)$ by the sub-basis $\Omega^s(M)$. In fact $\eta^s(M)$ is the collection U of all unions of finite intersections of elements of $\Omega^s(M)$ [6]. We call this topology the second classical Zariski topology of M.

Theorem 3. *Let R be a associative Noetherian ring and let M be a cotop multiplicational R-module with finite length. Assume that the second classical Zariski topology of M and the Zariski topology of M considered in [5] coincide. Then M is a comultiplication R-module.*

Theorem 4. *Let R be a associative Noetherian ring and let M be a co-multiplication R-module with finite length. Then $Spec^s(M)$ is a spectral space (with the second classical Zariski topology).*

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Incomplete spaces of idempotent measures

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The notion of idempotent measure in the idempotent mathematics (i.e., a part of mathematics dealing with idempotent operations on reals) corresponds to that of probability measure in the traditional mathematics [7]. The compact spaces of idempotent measures are intensively investigated by numerous authors. We are going to present some results on topological and categorical properties of the idempotent measures in the noncompact case.

One defines the set of idempotent measures on X as the set of functionals $\mu\colon C(X,[0,1]) \to [0,1]$ satisfying: 1) μ preserves the constants; 2) $\mu(t\phi) = t\mu(\phi)$; 3) $\mu(\phi \vee \psi) = \mu(\phi) \vee \mu(\psi)$.

The set $I(X)$ of idempotent measures is endowed with the weak* topology.

Note that exists a natural (by X) map $\xi_X\colon I^2(X) \to I(X)$ defined as follows. Given $\phi \in C(X,[0,1])$, let $\bar{\phi}\colon I(X) \to [0,1]$ be the function defined by $\bar{\phi}(\mu) = \mu(\phi)$, $\mu \in I(X)$. Then, for any $M \in I^2(X)$, $\xi_X(M)(\phi) \overset{def}{=} M(\bar{\phi})$.

It turns out that the definition of idempotent measure can be formulated in terms of special subsets of $X \times [0,1]$. Namely, we consider subsets $A \subset X \times [0,1]$ of the form: A is closed and saturated; $X \times \{0\} \subset A$; $A \cap (X \times \{1\}) \neq \emptyset$.

To every such set A there corresponds a functional $A : C(X,[0,1]) \to [0,1]$ (we thus keep the same notation) defined by the formula: $A(\phi) = \sup\{t\phi(x) \mid (x,t) \in A\}$.

Let X be a Tychonov space (completely regular space). By βX we denote the maximal compactification (Stone-Čech compactification) of X. Consider the set $I_\beta(X)$ of all subsets A in $X \times [0;1]$ such that :

1) A is closed in $X \times [0,1]$;
2) A is saturated, i.e. $\forall (x,t) \in A\ \forall t', 0 \le t' \le t,\ (x,t') \in A$;
3) $X \times \{0\} \subset A$;
4) the support of A, i.e., the set $supp(A) = \overline{\{x \in X | \exists t > 0, (x,t) \in A\}}$ is compact;
5) $\exists x \in X\colon (x,1) \in A$.

We denote by $I_\omega(X)$ the family of all sets $A \in I_\beta(X)$ such that $supp(A)$ is a finite set. Now we define map $\xi : I^2_\omega(X) \to I_\omega(X)$, where ξ

$\xi(A) = \{\ (x,r)|\ \exists\ s,t \in [0,1],\ \alpha \in I_\omega(X)$ such that $r = st, (x,s) \in \alpha,\ (\alpha,t) \in A\ \}$.

Next, we consider the case of metric spaces. For given metric space (X,d), we endow $X \times [0,1]$ by the metric $\hat{d}$, where $\hat{d}((x,t),(x',t')) = d(x,x') \vee |t-t'|$. The space $I_\omega(X)$ is endowed with Hausdorff metric induced by $\hat{d}_H$. We can consider $I_\omega(X)$, as a new metric space with Hausdorff metric d_H. Apply the same operation I_ω to $(I_\omega(X), d_H)$. We obtain a new space $I_\omega(I_\omega(X))$ with (d_{HH}) metric.

Theorem 1. *The map ξ_X: $(I^2_\omega(X), d_{HH}) \to (I_\omega(X), d_H)$ is non-expanding.*

This theorem allows us to extend the map ξ_X over the completion $\overline{I}(X)$ of $I_\omega(X)$ (here we assume that X is complete).

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Hyperspaces of convex sets related to idempotent mathematics

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The notion of hyperspace is one of the most important not only in topology but also in another parts of mathematics. This notion allows us to consider multivalued maps as single valued with the values being points of a hyperspace.

The hyperspace of compact convex sets in compact convex subsets of Euclidean spaces was first considered by Nadler, Quinn, and Stavrokas [4]. They proved, in particular, that the hyperspace of the Euclidean space $\mathbb{R}^n$, $n \geq 2$, is homeomorphic to the punctured Hilbert cube.

We denote by $x \oplus y$ the coordinatewise maximum of $x, y \in \mathbb{R}^n$. Given $t \in \mathbb{R}$ and $(y_1, \dots, y_n) \in \mathbb{R}^n$, let $t \otimes (y_1, \dots, y_n) = (\min\{t, y_1\}, \dots, \min\{t, y_n\})$. A subset $A \subset \mathbb{R}^n$ is said to be max-min convex if, for any $x, y \in A$ and any $t \in \mathbb{R}$, we have $x \oplus t \otimes y \in A$. It is proved in [3] that the hyperspace of compact max-min convex sets in the Euclidean space $\mathbb{R}^n$, $n \geq 2$, is homeomorphic to the punctured Hilbert cube.

Following the style of idempotent mathematics we define, for any $t \in \mathbb{R}$ and any $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $t \odot x = (t + x_1, \dots, t + x_n)$. A subset $A \subset \mathbb{R}^n$ is said to be max-plus convex (see, e.g., [1]) if, for any $x, y \in A$ and any $t \in (-\infty, 0]$, we have $x \oplus t \odot y \in A$. It is proved in [3] that the hyperspace of compact max-min convex sets in the Euclidean space $\mathbb{R}^n$, $n \geq 2$, is homeomorphic to the punctured Hilbert cube.

Recall that the Fell topology on the hyperspace of closed subsets of a Hausdorff topological space has as a subbase all sets of the form $\{A \mid A \cap V \neq \emptyset\}$, where V is an open subset of X, and also all sets of the form $\{A \mid A \subset W\}$, where W has compact complement. We denote by $\mathrm{Mpcc}_F\mathbb{R}^n$ and $\mathrm{Mmcc}_F\mathbb{R}^n$ the hyperspaces of the max-plus convex and max-min convex nonempty closed (not necessarily bounded) subsets of $\mathbb{R}^n$ endowed with Fell topology. See [5] for description of topology of the hyperspaces of compact convex subsets of $\mathbb{R}^n$ endowed with Fell topology.

Every non-empty closed subset A of a metric space (X, d) can be identified with the distance function $x \mapsto d(x, A)$. The topology of convergence on bounded sets induces the Attouch-Wetts topology on the hyperspace of non-empty closed sets. We denote by $\mathrm{Mpcc}_{AW}\mathbb{R}^n$ and $\mathrm{Mmcc}_{AW}\mathbb{R}^n$ the hyperspaces of the max-plus convex and max-min convex nonempty closed (not necessarily bounded) subsets of $\mathbb{R}^n$ endowed with Attouch-Wetts topology. Some results on ANR-properties of the hyperspaces in the Attouch-Wetts topology can be found in [6].

The aim of the talk is to discuss properties of the hyperspaces $\mathrm{Mpcc}_F\mathbb{R}^n$ and $\mathrm{Mmcc}_F\mathbb{R}^n$, $\mathrm{Mpcc}_{AW}\mathbb{R}^n$, and $\mathrm{Mmcc}_{AW}\mathbb{R}^n$.

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Volumes of knots and links in spaces of constant curvature

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We investigate the existence of hyperbolic, spherical or Euclidean structure on cone-manifolds whose underlying space is the three-dimensional sphere and singular set is a given knot or link. For two-bridge knots with not more than seven crossings we present trigonometrical identities involving the lengths of singular geodesics and cone angles of such cone-manifolds. Then these identities are used to produce exact integral formulae for the volume of the corresponding cone-manifold modeled in the hyperbolic, spherical and Euclidean geometries.

Twistor spaces on foliated manifolds

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Let M^{2n} be an even-dimensional Riemannian manifold, the twistor space $Z(M)$ is the parametrizing space for compatible almost complex structures on M. It is a bundle over M, with fiber $SO(2n)/U(n)$ and is equipped with two almost complex structures $J^{\pm}$, where J^{+} can be integrable but J^{-} is never integrable, however, it still is important as will be discussed. Moreover, in the case where J^{+} is integrable, it is shown in [1] that M has particular properties, especially when $n = 2$, which is an interesting case in physics, since the holomorphic structure of the twistor space correspond to a conformal structure of M. This correspondence is called the Penrose correspondence.

This talk is based on a joint work with R. A. Wolak [2] , in which, the theory of twistors on foliated manifolds is developed. We construct the twistor space of the normal bundle of a foliation. It is demonstrated that the classical constructions of the twistor theory lead to foliated objects and permit to formulate and prove foliated versions of some well-known results on holomorphic mappings. Since any orbifold can be understood as the leaf space of a suitably defined Riemannian foliation we obtain orbifold versions of the classical results as a simple consequence of the results on foliated mappings.

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Two problems in nonholonomic geometry (in quest of a co-worker)

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Particular distributions living on the stages of the Goursat Monster Tower (GMT) have been the object of studies dating as far back as 1889, then 1896, 1914, 1922, 1978, 1982, 1999, 2001 ... (the very notion GMT is, of course, not that old). From the GMT stage No 8 onwards there exist local numerical invariants (moduli) in the local classification of Goursat distributions (cf. Travaux en cours **62**, Paris (2000), Remarks 3 and 4 on p. 110 and 111).

A. A. Agrachev asked in the year 2000 if the moduli of Goursat structures descend to the level of the nilpotent approximations (NA) – simpler objects retaining some basic properties of initial completely nonholonomic distributions. (The NA's are of central importance in the geometric control theory, motion planning problems, etc.) The author gave two very partial answers to Agrachev's question (in 2006 and 2014, both in the negative). Otherwise, this **problem** remains vastly open.

The talk's proper aim is to survey another **problem** concerning the GMT. The one which asks for all the strongly nilpotent points (or, better: strata) in the stages of the GMT. (In the way of explanation, all points in the GMT are weakly nilpotent in the control theory sense, while only a tiny portion of them is strongly nilpotent.) The conjecture, still unsettled, says that, within the GMT, 'strongly nilpotent' is but a synonym of 'tangential', while all the tangential points are known [already since the mid 2000s] to be strongly nilpotent. So the brunt of this problem boils down to the computation of the NA's at non-tangential points. That little or ... that much.

To just give a non-trivial example, here is a non-tangential stratum RRVRV lying in the 5th stage of the GMT. The associated weights, central in the nonholonomic geometry and analysis, are 1, 1, 2, 3, 5, 7, 11. The NA(RRVRV) computed along the lines of the by-now-classical Bellaiche algorithm is — in certain *adapted* coordinates $z_1, z_2, \ldots, z_7$ — spanned by the two vector fields $\partial/\partial z_1$ and

$$\partial/\partial z_2 + z_1\partial/\partial z_3 + z_1z_2\partial/\partial z_4 + z_1z_4\partial/\partial z_5 + z_1z_3z_4\partial/\partial z_6 + \Big(z_1z_3z_6 + \frac{1}{3}z_1z_3^3z_4\Big)\partial/\partial z_7\,.$$

Yet, in this line of research *super*-adapted coordinates are sought, in which a visualisation of a given NA uses as few active variables as possible. In the chosen example such super-coordinates can be derived from the previous ones. The eventual visualisation of the NA(RRVRV) appears to be

$$\Big(\partial/\partial z_1\,, \qquad \partial/\partial z_2 + z_1\partial/\partial z_3 + z_1z_2\partial/\partial z_4 + z_1z_2z_3\partial/\partial z_5 + z_1z_2z_3^2\partial/\partial z_6 + z_1z_2z_3^4\partial/\partial z_7\Big)$$

(it is not possible to visualise NA(RRVRV) in only two active adapted variables; three as in the expansion above is the minimal number). Only having NA(RRVRV) in this utmostly compactified form, it becomes possible to show that the stratum RRVRV is indeed not strongly nilpotent.

The outlined problem is, therefore, pretty much computational. A skilful computer-oriented person is sought in earnest, willing to actively take part in dealing with this challenging problem.

The local τ-density of a linearly ordered spaces

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A set $A \subset X$ is said to be dense (in X), if $[A] = X$. The density of the space X is defined as the smallest cardinal $|A|$, where A is a dense subset of X [1]. This cardinal is denoted by $d(X)$. If $d(X) = \tau$, $\tau \geq \aleph_0$, the space X is said to be $\tau-$dense. If $d(X) \leq \aleph_0$, then X is said to be separable.

A topological space X is called locally $\tau-$dense at the point $x \in X$, if τ is the smallest cardinal number, such that x has a neighbourhood of density τ in X [2]. Local density at x is denoted by $ld(x)$. Local density of the space X is defined as follows:

$$ld(X) = \sup\{ld(x) : \ x \in X\}.$$

It is clear that local density of a topological space cannot exceed the density of said space, i.e. $ld(X) \leq d(X)$.

We say that the weak density of the topological space is $\tau \geq \aleph_0$, if τ is the smallest cardinal number such that there exists a π-base coinciding with τ of centered systems of open sets, i.e. there is a π-base $B = \cup\{B_\alpha : \alpha \in A\}$ where B_α is a centered system of open sets for each $\alpha \in A, |A| = \tau$ [3]. Weak density of topological space X is denoted by $wd(X)$.

Topological space X is said local weak τ-dense at a point x, if τ is the smallest cardinal number such that x has a neighborhood of weak density τ in X [4]. Local weak density at a point x is denoted by $lwd(x)$. The local weak density of a topological space X is defined as the supremum of all numbers $lwd(x)$ for $x \in X$:

$$lwd(X) = sup\{lwd(x) : x \in X\}.$$

It is clear that local weak density of a topological space cannot exceed the weak density of said space, i.e. $lwd(X) \leq wd(X)$.

Let X be a set, and $<$ be some relation on X. We say that $<$ is a linear order on X if the relation $<$ satisfies the following properties:

1) If $x < y$ and $y < z$, then $x < z$;
2) If $x < y$ then the relation $y < x$ does not hold;
3) If $x \neq y$ then either $x < y$ or $y < x$ holds.

A set X together with some linear order defined on it is called a linearly ordered set [1].

Theorem 1. *Suppose that a space X satisfies at least one of the following conditions:*

1) X is a linearly ordered topological space with the interval topology,
2) X is pseudometric space.

Then X is locally $\tau-$dense if and only if it is locally weak $\tau-$dense.

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Entropy and phase transitions in Calabi-Yau space

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The evolution of the concept of entropy from the mathematical definition of the theory of probability to the physical definition of entropy in various systems is considered. Statistical interpretation of entropy for a macrostate is characterized by N_i microstates

$$W = \frac{N}{N_1 N_2 \ldots} = \frac{N}{\Pi_i N_i}.$$

The equilibrium state corresponds to the maximum probability, which is proportional to the maximum entropy, and is displayed by the Boltzmann-Planck law

$$S = k \cdot lnW.$$

For large N using the Stirling formula

$$lnN! = NlnN - N$$

and taking into account the degeneracy of the energy states, z_i, we have the following formula for the probability of a system of particles

$$W = \frac{N}{n_1 n_2 \ldots n_r} \Pi_i z_i^{n_i}.$$

Within the framework of the AdS/CFT correspondence, a model for determining the entropy of black holes through the number of microstates is considered. It is known that the entropy of a black hole is determined by the Bekenstein-Hawking formula,

$$S = \frac{A}{4G},$$

where A and G are the surface areas of the black hole and the gravitational constant, respectively. In the framework of superstring and D-brane theory, the concept of entropy has changed due to the presence of the extra-dimensional Calabi-Yau space, which is folded at each point of the usual Minkowski space

$$S = \frac{A_{d+p}}{4G_{d+p}},$$

where p are spatial directions of the space $R^p \times S^{d-1}$, with d – the number of space dimensions transverse to the p-directions. According to Strominger [1], a black hole can be represented as a submanifold of such a Calabi-Yau like a pea in a shell. Depending on the dimension of space, the two-dimensional world surface of the string completely surrounds the two-dimensional sphere, the 3-brane surrounds the three-dimensional sphere, etc. Since the black hole tends to deflate and swell, according to the ideology of flop transformations, a rupture of the Calabi-Yau space occurs. In this case, according to Strominger's calculations, the black hole undergoes a phase transition and transforms into a pointlike particle like a photon,

$$\frac{S_{array}}{S_{string}} \sim \left(\frac{R}{r_H}\right)^{1/(d-3)}.$$

So the array dominates for small horizon radii, and the black string dominates for large horizon radii.

String theory spacetimes with conserved quantum numbers can be black holes, but more commonly they are black p-branes. According to papers [2, 3] black hole entropy

$$S_{BH} = \frac{\Omega_{d-2} r_H^{d-2}}{4G_d}$$

can be described in terms of D-brane theory,

$$S_{BH} = \frac{\Omega_{8-p} r_H^{8-p}}{4G_{10-p}} cosh\beta,$$

where $cosh\beta$ depends on the number of branes. For particular cases, when the number of branes that cover a black hole is determined, we can calculate $d = 4$ entropy in usual four-dimensional space-time. D-brane method for a microscopic accounting for S_{BH} of BPS black holes with macroscopic entropy leads to the formula

$$S_{BH} = 2\pi\sqrt{N_2 N_6 N_5 N_m},$$

(N_i - the numbers of i-branes) which is in agreement with black hole entropy formula, [2].

Using BPS - states of D-branes represented by vector bundles of the type

$$\begin{array}{c} Spin(k) \to Spin(k+1) \\ \downarrow \\ S^k \end{array} \tag{1}$$

it can be shown that for $k = 6$, $Spin(6)$ group is isomorphic to the $SU(4)$ group. Since the group describing black holes is $SU(2,2|4) \sim SU(2,2) \times SU(4)$ ($SU(2,2)$ describes the external degrees of freedom, and $SU(4)$ - the internal ones), the greatest interest is of group $SU(4)$. Then we can work with $Spin$ vector bundles, which present D-branes with the phase transitions between them classified with Grothendieck K-group in the framework of the Clifford algebra formalism. As a result, we obtain a chain of phase transitions of a black hole represented by transitions between topological invariants of vector bundles described by K-groups

$$K(S^6) \to K(S^4) \to K(S^2) \to K(S^0) = Z \quad ,$$

which signal about an equidistant set of energy levels of a point-like particle into which the black hole has passed during the phase transition.

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Reducing singularities of smooth functions to normal forms

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The talk is devoted to the "uniform" reduction of C^∞ smooth functions on 2-dimensional manifolds to a canonical form at the singular points of these functions.

Definition 1. A smooth function $f = f(u_1, u_2)$ has a singularity E_6 at its critical point $P \in \mathbb{R}^2$ if

(i) the first and second differentials $df(P) = 0$, $d^2 f(P) = 0$, and the third differential $d^3 f(P) \neq 0$ and is a perfect cube;

(ii) there exists a vector $v \in \operatorname{Ker} d^3 f(P)$ such that $v^4 f \neq 0$ (by $v^4 f$ we mean the fourth derivative of f along the tangent vector v at P).

Theorem 2 (Reducing E_6 to normal form)**.** *Let the function $f(u_1, u_2)$ have a singularity E_6 at the critical point P. Then, in some neighborhood of P, there is a local coordinate system in which the point P is the origin, and the function has the normal form*

$$f = f(P) + \tilde{x}^3 \pm \tilde{y}^4.$$

Moreover, this coordinate system can be chosen in such a way that the coordinate change $(u_1, u_2) \to (\tilde{x}, \tilde{y})$ can be expressed in terms of the original function and its partial derivatives of order ≤ 7 using algebraic operations and the operation of taking a proper integral.

Remark 3. In [1], the first part of Theorem 2, i.e. the existence of a coordinate change, was proved using Tougeron's theorem [2]; in view of this, obtaining a formula for the corresponding coordinate change requires solving the Cauchy problem for a system of ordinary differential equations. We construct our coordinate change explicitly, without using Tougeron's theorem.

Our proof of Theorem 2 consists of three steps. At first, we consider a smooth function in two variables of the form $f(u_1, u_2) = u_1^3 + u_2^4 + R(u_1, u_2)$, where the Taylor series of the function $R(u_1, u_2)$ at the origin has zero coefficients at all monomials of the form $u_1^k u_2^l$, $4k + 3l \leq 12$. At second, by the sequence $(u_1, u_2) \to (x, y)$ of substitutions described in [1], we reduce the function f to such a form that the origin leaves fixed and $f^{(k+l)}_{x^k y^l}(0,0) = 0$, where $0 \leq k < 3, 0 \leq l < 4$. At third, using the Taylor series expansion of the function with an integral remainder, we reduce the function to the required normal form by the coordinate change

$$\phi : \mathbb{R}^2_{x,y} \to \mathbb{R}^2_{\tilde{x},\tilde{y}} \qquad \text{with} \quad \tilde{x} = x\sqrt[3]{g(x,y)}, \quad \tilde{y} = y\sqrt[4]{h(x,y)},$$

where $g(x,y) := \sum_{k=0}^{3} \frac{y^k}{k!} f^{(k)}_{y^k}(x,0)/x^3$ and $h(x,y) := \frac{1}{6}\int_0^1 f^{(4)}_{y^4}(x, sy)(1-s)^3\, ds$.

Our next goal is to describe a neighborhood in which the above coordinate change ϕ exists and is regular. To simplify our computations, we will assume that the following is true:

Assumption 4. $f^{(4)}_{y^4}(0,0) = 24$, $f'''_{x^3}(0,0) = 6$.

Theorem 5 (Estimating the radius of a neighborhood where the coordinate change is regular)**.** *Under the hypotheses of Theorem 2 and Assumption 4, let $U_0 = \{(x,y) \mid \max(|x|,|y|) < R_0\}$ be a neighborhood of the origin such that $C_{\alpha\beta} = \sup_{U_0} \left| f^{\alpha+\beta}_{x^\alpha y^\beta}(x,y) \right| \leq M$ for $(\alpha, \beta) \in \{(0,5), (1,4), (3,1), (3,2), (3,3), (4,0), (4,1), (4,2), (4,3)\}$, where $R_0 > 0$, $M \geq 0$. Let's consider the neighborhood $U = \{(x,y) \mid \max(|x|,|y|) < R\}$, where the positive constant R is defined by the formula: $R = \min\{R_0, \frac{2}{M+2}\}$. Let also ϕ be the coordinate change from the proof of Theorem 2 that reduces f to the normal form E_6. Then:*

(a) *the functions $h(x,y)$ and $g(x,y)$ do not change sign in U, that is, the change $\phi|_U$ is well defined and is smooth;*
(b) *at every point $\boldsymbol{x} \in U$, one has $\|\phi'(\boldsymbol{x}) - I\| < C < 1$, where $C = \frac{2}{5}$, i.e., the coordinate change $\phi|_U$ is close to the identity;*
(c) *the coordinate change $\phi|_U$ is injective and regular, i.e., it is an embedding and* $\det|\phi'(\boldsymbol{x})| \neq 0$ *at every point $\boldsymbol{x} \in U$, moreover the image of this embedding contains the disk of radius $(1-C)R$ centred at the origin.*

Remark 6. Our coordinate change $\phi|_U$ from Theorem 2 and Theorem 5 provides a "uniform" reduction of the function f at a singular point of type E_6 to the canonical form $\tilde{x}^3 \pm \tilde{y}^4$ in the sense that the neighbourhood radius and the coordinate change we constructed in this neighbourhood (as well as all partial derivatives of the coordinate change) continuously depend on the function f and its partial derivatives. A uniform reduction of smooth functions near critical points to a canonical form was known earlier for the case of smoothly stable singularities [3]. A uniform reduction of smooth functions to a canonical form by C^k-smooth changes (for finite $k < \infty$) is known for finite type singularities [3] and for topologically stable singularities [5].

To prove Theorem 5, we apply the following lemma to the coordinate transformation ϕ from the proof of Theorem 2. We estimate the norm of the matrix in terms of its elements: $\|A\| \leq \sqrt{\sum a_{ij}^2}$, $A = \{a_{ij}\}_{i,j=1}^n$.

Lemma 7. *Let $\phi : U \to \mathbb{R}^n$ be a smooth mapping, where U is a convex open subset of $\mathbb{R}^n$. Let the differential of ϕ have the form $\phi'(\mathrm{x}) = I + A(\mathrm{x})$, where I is the unit matrix of dimension n, $\|A\| < C$, $0 < C < 1$. Then ϕ is injective and* $\det|\phi'(\mathrm{x})| \neq 0$ *at every point $\mathrm{x} \in U$, i.e., ϕ is a diffeomorphism to its image $\phi(U)$. Moreover, $\langle \phi'(\mathrm{x}), \mathrm{x} \rangle \geq (1-C)|\mathrm{x}|^2$ for every point $\mathrm{x} \in U$.*

From the last assertion of Lemma 7 and [6, Corollary 8.3, Step 1], we conclude that $\phi(U)$ contains the disk of radius $(1-C)R$ centred at the origin. This completes our proof of Theorem 5.

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On m-convexity and m-semiconvexity of sets in Euclidean spaces

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The topological and geometric properties of classes of generally convex sets in multidimensional real Euclidean space $\mathbb{R}^n$, $n \geq 2$, known as m-convex, weakly m-convex, m-semiconvex, and weakly m-semiconvex, $m = 1, 2, \ldots, n-1$, are studied in [1]–[6]. A set of the space $\mathbb{R}^n$ is called ***m*-convex** (***m*-semiconvex**) if for any point of the complement of the set to the whole space there is an m-dimensional plane (half-plane) passing through this point and not intersecting the set. An open set of the space is called **weakly *m*-convex** (**weakly *m*-semiconvex**), if for any point of the boundary of the set there exists an m-dimensional plane (half-plane) passing through this point and not intersecting the given set. A closed set of the space is called **weakly *m*-convex** (**weakly *m*-semiconvex**) if it is approximated from the outside by a family of open weakly m-convex (weakly m-semiconvex) sets. These notions were proposed by Professor Yuri Zelinskii [1], [2].

Let us denote the classes of m-convex and weakly m-convex sets in $\mathbb{R}^n$, $n \geq 2$, by $\mathbf{C^n_m}$ and $\mathbf{WC^n_m}$, respectively. There are weakly m-convex sets in $\mathbb{R}^n$, $n \geq 2$, $1 \leq m < n$, which are not m-convex, i. e., the class $\mathbf{WC^n_m} \setminus \mathbf{C^n_m}$ is not empty for any $m = 1, 2, \ldots, n-1$. The example of an open set of the class $\mathbf{WC^2_1} \setminus \mathbf{C^2_1}$ is constructed in [4]. The examples of open and closed sets of $\mathbf{WC^n_{n-1}} \setminus \mathbf{C^n_{n-1}}$ and examples of open sets of $\mathbf{WC^n_m} \setminus \mathbf{C^n_m}$, $n \geq 3$, $1 \leq m < n-1$, are constructed in [6]. Moreover, any open or compact set of $\mathbf{WC^n_{n-1}} \setminus \mathbf{C^n_{n-1}}$ is necessarily disconnected, but there exist domains of $\mathbf{WC^n_m} \setminus \mathbf{C^n_m}$, $n \geq 3$, $1 \leq m < n-1$, which show the following three theorems.

Theorem 1. ([4]) *An open set of the class $\mathbf{WC^n_{n-1}} \setminus \mathbf{C^n_{n-1}}$ consists of at least three connected components.*

Theorem 2. ([6]) *A compact set of the class $\mathbf{WC^n_{n-1}} \setminus \mathbf{C^n_{n-1}}$ consists of at least three connected components.*

Theorem 3. ([6]) *There exist domains of the class $\mathbf{WC^n_m} \setminus \mathbf{C^n_m}$, $n \geq 3$, $1 \leq m < n-1$.*

It is also known the topological classification of open (weakly) $(n-1)$-convex sets in the space $\mathbb{R}^n$ with smooth boundary [1], [4]. Each such a set is convex, or consists of no more than two unbounded connected components, or is given by the Cartesian product $E^1 \times \mathbb{R}^{n-1}$, where E^1 is a subset of $\mathbb{R}$.

Let us denote the classes of m-semiconvex and weakly m-semiconvex sets in $\mathbb{R}^n$, $n \geq 2$, by $\mathbf{S^n_m}$ and $\mathbf{WS^n_m}$, respectively. In [3] it is constructed an example of an open set of the class $\mathbf{WS^2_1} \setminus \mathbf{S^2_1}$. It is also conjectured that any open set of $\mathbf{WS^2_1} \setminus \mathbf{S^2_1}$ consists of at least three components. The latter statement is proved in [4]. There can be also constructed sets of $\mathbf{WS^n_{n-1}} \setminus \mathbf{S^n_{n-1}}$ and the example of domains of $\mathbf{WS^n_m} \setminus \mathbf{S^n_m}$, $n \geq 3$, $1 \leq m < n-1$, similar to the domains of $\mathbf{WC^n_m} \setminus \mathbf{C^n_m}$. The following theorem shows the impossibility of the topological classification of weakly 1-semiconvex sets with smooth boundary similar to the topological classification of open $(n-1)$-convex and weakly $(n-1)$-convex sets with smooth boundary.

Theorem 4. ([5]) *An open, bounded set of the class $\mathbf{WS^2_1} \setminus \mathbf{S^2_1}$ with smooth boundary consists of at least four connected components.*

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On representations of q_{ij}-commuting isometries

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C^*-algebras generated by isometries have been studied by various authors. Among the most relevant examples we mention Toeplitz algebras, Cuntz algebras, and their deformations. These examples belong to the class of $*$-algebras with Wick ordering [1].

Recall that the Cuntz-Toeplitz algebra $\mathcal{O}_d^0$ is a unital C^*-algebra generated by elements s_j, $j = 1, \ldots, d$, which satisfy relations

$$s_j s_k = \delta_{jk} I, \quad j, k = 1, \ldots, d.$$

In this paper, we consider representations of C^*-algebra W_d generated by elements s_j, $j = 1, \ldots, d$, satisfying relations

$$s_i^* s_i = I, \quad s_i^* s_j = q_{ij} s_j s_i^*, \quad |q_{ij}| < 1, \; q_{ij} = \bar{q}_{ji}, \qquad 1 \leq i \neq j \leq d. \tag{1}$$

One can see that for $q_{ij} = 0$, $i \neq j$, this algebra is $\mathcal{O}_d^0$. It was conjectured in [2] that, in particular, for $|q_{ij}| < 1$, $i \neq j$, the corresponding C^*-algebra is isomorphic to $\mathcal{O}_d^0$, however, the proof is known for the cases $d = 2$ [3] or $|q_{ij}| < \sqrt{2} - 1$ [2] only. While the representations of the Cuntz-Toeplitz algebras were studied in detail in a number of papers, for other Wick algebras, including W_d, only the Fock representation [1] is known. Therefore, constructing representations of "deformed" relations (1) can give a hint for a construction of the isomorphism between W_d and $\mathcal{O}_d^0$ in a general case.

We start with some notations. Let $\alpha = (\alpha_1, \ldots, \alpha_m) \in \{1, \ldots, d\}^m$ be a finite multiindex of length m, $|\alpha| = m$, let $\Lambda_m = \{1, \ldots, d\}^m$ be the set of all finite multiindices of length m, $\Lambda_0 = \emptyset$, and let $\Lambda^0 = \cup_{m=0}^{\infty} \Lambda_m$ be the set of all finite multiindices of arbitrary length. Also, we will use the set $\Lambda = \{1, \ldots, d\}^\infty$ of all infinite multiindices. For each finite multiindex $\alpha = (\alpha_1, \ldots, \alpha_m) \in \Lambda^0$ we use notation $s_\alpha = s_{\alpha_1} \ldots s_{\alpha_m}$. For a finite multiindex we use standard mappings:

$$\begin{aligned}
&\Lambda_m \ni \alpha = (\alpha_1, \ldots, \alpha_m) \mapsto \sigma(\alpha) = (\alpha_2, \ldots, \alpha_m) \in \Lambda_{m-1}, \\
&\Lambda_m \ni \alpha = (\alpha_1, \ldots, \alpha_m) \mapsto \sigma_j(\alpha) = (j, \alpha_2, \ldots, \alpha_m) \in \Lambda_{m+1}, \quad j = 1, \ldots, d.
\end{aligned}$$

The same mappings can be obviously defined for an infinite multiindex $\alpha \in \Lambda$.

If $\alpha \in \Lambda^0$ does not contain j, then (1) implies

$$s_j^* s_\alpha = q(j, \alpha) s_\alpha s_j^*, \quad q(j, \alpha) = q_{j\alpha_1} \ldots q_{j\alpha_m}.$$

If α contains j, then α can be represented as $\alpha = (\alpha' j \alpha'')$, where α' does not contain j, then

$$s_j^* s_\alpha = q(j, \alpha') s_{\alpha'} s_{\alpha''} = q(j, \alpha) s_{\alpha \setminus j}$$

(here and below, we denote by $\alpha \setminus j = (\alpha' \alpha'')$ multiindex obtained from α by removing the first occurrence of j, and set $q(j, \alpha) = q(j, \alpha')$ for convenience).

For infinite multiindices $\alpha, \beta \in \Lambda$, we define $q(\alpha, \beta)$ as follows. If there exists $\gamma \in \Lambda$, $\alpha', \beta' \in \Lambda_m$, $m \geq 0$, for which

$$\alpha = (\alpha'\gamma), \ \beta = (\beta', \gamma), \quad \alpha' \text{ and } \beta' \text{ coincide up to a permutation,}$$

then we define $q(\alpha, \beta) = q(\alpha', \beta')$, and zero otherwise. It is a straightforward fact that $q(\alpha, \beta)$ is well-defined.

We proceed with introducing an appropriate Hilbert space. We say that infinite multiindices $\alpha, \beta \in \Lambda$ are equivalent, denoted by $\beta \sim \alpha$, if they "have the same tails up to a shift", i.e., there exist numbers m, n, such that $\sigma^m(\alpha) = \sigma^n(\beta)$. Fix an infinite multiindex α and consider a family of vectors $(e_\beta \mid \beta \sim \alpha)$. For these vectors, define

$$(e_\beta, e_\gamma) = q(\beta, \gamma), \tag{2}$$

in particular, $(e_\beta, e_\beta) = 1$.

Proposition 1. *Form (2) is well-defined and positive.*

For a fixed $\alpha \in \Lambda$, define a Hilbert space H_α as the closed linear span of vectors $(e_\beta \mid \beta \sim \alpha)$ with respect to the introduced scalar product.

Theorem 2. *1. Operators in H_α*

$$\pi_\alpha(s_j)e_\beta = e_{\sigma_j(\beta)}, \quad \pi_\alpha(s_j^*)e_\beta = \begin{cases} 0, & \beta \textit{ does not contain } j, \\ q(j, \beta)e_{\beta \setminus j}, & \textit{otherwise,} \end{cases}$$

form well-defined $*$-representation of the C^*-algebra W_d.

2. This representation is irreducible

3. Representations corresponding to multiindices α, α' are unitary equivalent iff the corresponding Hilbert spaces coincide, i.e., $\alpha \sim \alpha'$.

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Homotopic Nerve Complexes with Free Group Presentations

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This paper introduces homotopic nerve complexes in a planar Whitehead CW space [6, §4-5] and their Rotman free group presentations [4, §11,p.239]. A CW complex in a space K is a closure-finite cell complex that is Hausdorff (union of disjoint cells), satifying the containment property (closure of every cell complex is in K) and intersection property (common parts of cell complexes in K are also in K). A complex K is locally finite, i.e., every point $p \in K$ is a member of some finite subcomplex of K and every complex has a finite number of faces [5, §5.2,p.65]. A planar CW complex K is a collection of 0-cells (vertexes), 1-cells (edges) and 2-cells (filled triangles). Collections of planar cells attached to each other are sub-complexes in K.

Definition 1. 1-Cycle. A 1-cycle cyc$E$ in a CW space K is a collection of path-connected vertexes on 1-cells (edges) attached to each other and with no end vertex.

The edges in a 1-cycle cell complex in a CW space are replaced by homotopic maps to obtain a homotopic cycle.

Definition 2. Homotopic-Cycle. A **homotopic cycle** E (denoted by cyc$E$) is defined to be $\{h_i\}_{i=1}^n$, a set of n paths in a space X, where $h_1(0) = h_n(1)$ and the initial point of h_{i+1} is the terminal point of h_i for $2 \leq i \leq n-1$, i.e., $h_i(0) = h_{(i-1)}(1)$. Each path is a mapping $h : [0,1] \to X$ and $h_i(0)$ is a vertex in a finite set of cycle vertices. A reverse path $\bar{h}_i(t) := h_i(t-1)$ gives us an inverse map, so that

$$h_i(0) - \bar{h}_i(1) = h_i(0) - h_i(1-1) = h_i(0) - h_i(0) = 0.$$

In cycle cyc$E$, every vertex v_i is reachable by k maps from a distinguished vertex $h_1(0) = v_0$, *i.e.*,

$$\begin{aligned} kv_0 &:= h_1(0) + \cdots + h_{k+1}(0) \\ &\overbrace{:= h_1 \to \cdots \to h_{k+1}}^{\textbf{\textit{i.e.}, } k \textbf{ maps to reach } h_{k+1}(0) \textbf{ from } h_1(0)}. \end{aligned}$$

Here, $+$ represents a move from one vertex to another one in the cycle, which translates to a homotopic path between vertices.

Definition 3. Nerve Complex. A nerve complex Nrv$E$ in a space X is a collection of nonempty cell complexes with nonvoid intersection.

Theorem 4. *A pair of pair of 1-cycles with a common vertex in a CW space is a nerve complex.*

Theorem 5. *Every collection of homotopic cycles with a common vertex in a CW space is a homotopic nerve complex.*

Lemma 6. *Every vertex in the triangulation of the vertices in a CW space is the nucleus of an Alexandroff-Hopf nerve complex* [1, §4.2.11, p. 161].

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Theorem 7. *A CW space containing n triangulated vertexes contains n Alexandroff-Hopf nerve complexes.*

Remark 8. A finite group G is free, provided every element $x \in G$ is a linear combination of its basis elements (called generators) [2, §1.4, p. 21]. We write $\mathcal{B}$ to denote a nonempty basis set of generators $\left\{g_1, \ldots, g_{|\mathcal{B}|}\right\}$ and $G(\mathcal{B}, +)$ to denote the free group with binary operation $+$.

Definition 9. Rotman Presentation[4, p.239] Let $X = \{g_1, \ldots\}, \triangle = \{v = \sum kg_i, v, g_i \in X\}$ be a set of generators of members of a nonempty set X and set of relations between members of X and the generators in X. A mapping of the form $\{X, \triangle\} \to G$, a free group, is called a presentation.

Definition 10. Let 2^K be the collection of cell complexes in a CW space K, $E \in 2^K$, basis $\mathcal{B} \in G$, k_i the i^{th} integer coeficient in a linear combination $\sum\limits_{i,j} k_i g_j$ of generating elements $g_j \in \mathcal{B}$. A free group G **presentation** of E is a continuous map $f : 2^K \to 2^K$ defined by

$$f(E) = \left\{ v := \sum_{i,j} k_i g_j : v \in E, g_j \in \mathcal{B}, k_i \in \mathbb{Z} \right\}$$
$$= \overbrace{\boldsymbol{G}(\left\{\boldsymbol{g_1}, \ldots, \boldsymbol{g_{|\mathcal{B}|}}\right\}, +)}^{\boldsymbol{E} \mapsto \textbf{free group } \boldsymbol{G}}.$$

Lemma 11. [3, §4, p. 10] *Every homotopic cycle in a space X has a free group presentation.*

Here are two main results.

Theorem 12. *Every homotopic cycle in a CW space has a free group presentation.*

Theorem 13. *Every homotopic nerve in a CW space has a free group presentation.*

Remark 14. An application of nerve complexes is given in terms of the approximation of video frame shapes.

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Uniform measures in Euclidean space

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A uniform measure in Euclidean space $\mathbb{R}^d$ is a measure that assigns to each ball $B(x,r)$ with center x in the support of the measure, a mass dependent of r and independent of the choice of x.

For example any invariant measure of a subgroup of the isometry group of $\mathbb{R}^d$ is uniform, and this sub-class of uniform measures are called homogeneous measures. There are known a few examples of non-homogeneous uniform measures, such as the volume measure of the "light cone" $\{x^2+y^2+z^2 = w^2\} \subset \mathbb{R}^4$.

The study of uniform measures in Euclidean space was initiated by David Preiss as the crucial ingredient of his 1987 proof of the Besicovitch conjecture [4], and one motivation for extending this study is to understand the structure of measures in general geometry. It is known (see [1]) that a uniform measure must be a multiple of the k-dimensional area measure restricted to a k-dimensional analytic variety, and the classification of k-dimensional uniform measures remains a difficult open problem, still open even in the plane (see also [2], [3]). I will present a classification [5] of 1-dimensional uniform measures in $\mathbb{R}^d$, and mention some open questions for more general dimensions. This is joint work with Paul Laurain, from Paris 7 University.

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Centralizers of elements in Lie algebras of vector fields

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Let $\mathbb{K}$ be an algebraically closed field of characteristic zero and $A = \mathbb{K}[x_1, \ldots, x_n]$ the polynomial ring over $\mathbb{K}$. A $\mathbb{K}$-derivation D of A is a $\mathbb{K}$-linear mapping $D\colon A \to A$ that satisfies the rule: $D(ab) = D(a)b + aD(b)$ for all $a,\ b \in A$. If $\mathbb{K} = \mathbb{R}$ then every derivation D on $\mathbb{R}[x_1, \ldots, x_n]$ can be considered as a vector field on $\mathbb{R}^n$ with polynomial coefficients. The vector space $W_n(\mathbb{K})$ (over the field $\mathbb{K}$)of all $\mathbb{K}$-derivations (or vector fields) on the polynomial ring A is a Lie algebra over $\mathbb{K}$. Any derivation $D \in W_n(\mathbb{K})$ can be uniquely extended on the field $R = \mathbb{K}(x_1, \ldots, x_n)$ of rational functions in n variables by the rule: $D(a/b) = (D(a)b - aD(b))/b^2$, the vector space $\widetilde{W_n}(\mathbb{K})$ of all derivations on R is also a Lie algebra, it is in fact the Lie algebra of all vector fields with rational coefficients on $\mathbb{K}^n$.

Recall that for a given Lie algebra L and its element $x \in L$ the set $C_L(x) = \{y \in L :\ [x, y] = 0\}$ is called the centralizer of x in L. The centralizer $C_L(x)$ is a subalgebra of the Lie algebra $W_n(\mathbb{K})$ containing the element x. The structure of centralizers of polynomial derivations is of significant importance due to applications in differential equations and geometry (see, for example [1], [2]).

Let p and q be algebraically independent irreducible polynomials from the ring A. A polynomial $f \in A$ will be called p-q-free if f is not divisible by any homogeneous polynomial in p and q of positive degree. One can write every polynomial $g \in A$ in the form g_0g_1, where g_0 is a p-q-free polynomial and $g_1 = h(p, q)$ for some homogeneous polynomial $h(s, t) \in \mathbb{K}[s, t]$. The (total) degree of h in s, t will be called the p-q-degree of g and denoted by $\deg_{p-q} g$. The following result gives a characterization of a centralizer of a polynomial derivation if its field of constants (in the field of rational functions) satisfies certain restrictions:

Theorem 1. *Let $D_1 \in W_n(\mathbb{K})$ be such a derivation that its field of constants* $\mathrm{Ker} D_1$ *in the field of rational functions $\mathbb{K}(x_1, \ldots, x_n)$ is of transcendence degree one and contains no nonconstant polynomials. Then* $\mathrm{Ker} D_1 = \mathbb{K}(\frac{p}{q})$ *for some irreducible and algebraically independent polynomials $p, q \in \mathbb{K}[x_1, \ldots, x_n]$ and $D_1 = hf(p, q)D_0$ for some homogeneous polynomial f in the variables p, q, p-q-free polynomial h and irreducible derivation D_0. Further, the centralizer $C = C_{W_n}(D)$ is one of the following Lie algebras: (1) $C = \mathbb{K}[p, q]_m hD_0$, where $\mathbb{K}[p, q]_m$ is the linear space of homogeneous polynomials in p, q and $m = \deg_{p\text{-}q} f$, (2) $C = (\mathbb{K}(\frac{p}{q})D_1 + \cdots + \mathbb{K}(\frac{p}{q})D_k) \cap W_n(\mathbb{K})$ for some linearly independent with D_1 derivations $D_2, \ldots, D_k \in C$ over the field $\mathbb{K}(x_1, \ldots, x_n)$.*

Moreover, C is of finite dimension over the field $\mathbb{K}$.

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Properties of quasisymmetric mappings to preserve the structures of spaces

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The class of quasisymmetric mappings on the real axis was first introduced by A. Beurling and L. V. Ahlfors [1]. Later P. Tukia and J. Väisälä [2] considered these mappings between general metric spaces. See, e.g., [3] for an overview of the results in this direction. In our work we generalize the concept of quasisymmetric mappings to the case of general semimetric spaces. We establish conditions under which the image $f(X)$ of a semimetric space X with the triangle function Φ_1 under η-quasisymmetric embedding f is a semimetric space with another triangle function Φ_2. Condition under which f preserves a Ptolemy inequality is also found as well as condition under which f preserves a relation "to lie between" imposed on three different points of the space.

Let X be a nonempty set. Recall that a mapping $d\colon X \times X \to \mathbb{R}^+$, $\mathbb{R}^+ = [0, \infty)$ is a *metric* if for all $x, y, z \in X$ the following axioms hold: (i) $(d(x, y) = 0) \Leftrightarrow (x = y)$, (ii) $d(x, y) = d(y, x)$, (iii) $d(x, y) \leqslant d(x, z) + d(z, y)$. The pair (X, d) is called a *metric space*. If only axioms (i) and (ii) hold then the pair (X, d) is called a *semimetric space*.

Definition 1. Let (X, d), (Y, ρ) be semimetric spaces. We shall say that an embedding $f\colon X \to Y$ is *η-quasisymmetric* if there is a homeomorphism $\eta\colon [0, \infty) \to [0, \infty)$ so that

$$d(x, a) \leqslant t d(x, b) \text{ implies } \rho(f(x), f(a)) \leqslant \eta(t)\rho(f(x), f(b))$$

for all triples a, b, x of points in X and for all $t > 0$.

A definition of a triangle function was introduced by M. Bessenyei and Z. Páles in [4].

Definition 2. Consider a semimetric space (X, d). We say that $\Phi\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a *triangle function* for d if Φ is symmetric and monotone increasing in both of its arguments, satisfies $\Phi(0, 0) = 0$ and, for all $x, y, z \in X$, the following generalized triangle inequality holds:

$$d(x, y) \leqslant \Phi(d(x, z), d(y, z)).$$

The most important triangle functions $\Phi(u, v)$ which generate well-known types of metrics and their generalizations are $u + v$ (metric), $K(u + v)$ (b-metric with $K \geqslant 1$), $\max\{u, v\}$ (ultrametric).

Proposition 3. *Let (X, d) be a semimetric space with the triangle function Φ_1, (Y, ρ) be a semimetric space and let $f\colon X \to Y$ be a surjective η-quasisymmetric embedding. Suppose that the following conditions hold for Φ_1 and for some function $\Phi_2\colon \mathbb{R}_+^2 \to \mathbb{R}_+$:*

(i) *Φ_2 is symmetric, monotone increasing in both of its arguments and satisfies $\Phi(0, 0) = 0$,*
(ii) *$\lambda\Phi_1(x, y) \leqslant \Phi_1(\lambda x, \lambda y)$ and $\Phi_2(\lambda x, \lambda y) \leqslant \lambda\Phi_2(x, y)$ for every $\lambda > 0$,*
(iii) *For every $t_1, t_2 \in \mathbb{R}_+ \setminus \{0\}$ the inequality*

$$1 \leqslant \Phi_1\left(\frac{1}{t_1}, \frac{1}{t_2}\right) \;\; \textit{implies} \;\; 1 \leqslant \Phi_2\left(\frac{1}{\eta(t_1)}, \frac{1}{\eta(t_2)}\right). \tag{1}$$

Then Φ_2 is a triangle function for the space (Y, ρ).

In what follows under Ptolemaic spaces we understand semimetric spaces (X, d) for which the well-known Ptolemy inequality

$$d(x, z)d(t, y) \leqslant d(x, y)d(t, z) + d(x, t)d(y, z)$$

holds. Note that this inequality does not imply the standard triangle inequality in (X, d).

Proposition 4. *Let (X, d) be a Ptolemaic space, (Y, ρ) be a semimetric space and let $f\colon X \to Y$ be a surjective η-quasisymmetric embedding. If for every $t_1, t_2, t_3, t_4 \in \mathbb{R}^+$ the inequality*

$$t_1t_2t_3t_4 \leqslant t_1t_2 + t_3t_4 \ \text{ implies } \ \eta(t_1)\eta(t_2)\eta(t_3)\eta(t_4) \leqslant \eta(t_1)\eta(t_2) + \eta(t_3)\eta(t_4), \tag{2}$$

then (Y, ρ) is also Ptolemaic.

Let (X, d) be a semimetric space and let x, y, z be different points from X. We shall say that the point y lies between x and z if the equality $d(x, z) = d(x, y) + d(y, z)$ holds. K. Menger [5] seems to be the first who formulated the concept of "metric betweenness" for general metric spaces.

Theorem 5. *Let (X, d), (Y, ρ) be semimetric spaces and let $f\colon X \to Y$ be η-quasisymmetric embedding. If the homeomorphism η has the form*

$$\eta(t) = \begin{cases} \frac{1}{2} + \Psi_1(t, 1-t), & t \in [0, 1], \\ \frac{1}{\frac{1}{2} + \Psi_2\left(\frac{1}{t}, 1 - \frac{1}{t}\right)}, & t \in [1 + \infty), \end{cases} \tag{3}$$

where Ψ_1, Ψ_2 are some continuous, antisymmetric, strictly increasing by the first variables, defined on $[0, 1] \times [0, 1]$ functions of two variables such that $\Psi_1(1, 0) = \Psi_2(1, 0) = 1/2$, then f preserves metric betweenness.

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IW contractions and their generalizations

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Contractions of Lie algebras are a kind of limit processes between orbits of such algebras. In 1953, Inönü and Wigner studied special contractions of Lie algebras as a part of broader study of contractions of Lie groups and their representations. These contractions were generalized by Doebner and Melsheimer in 1967. A rigorous general definition of contractions of Lie algebras was given by Saletan in 1961. He also studied contraction whose matrices are first-order polynomials with respect to contraction parameters. Since then, a number of conjectures about various ways of realizing contractions of Lie algebras had accumulated in the literature.

Definition 1. Let V be an n-dimensional vector space over $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$, $n < \infty$, and let $\mathcal{L}_n = \mathcal{L}_n(\mathbb{F})$ denote the set of all possible Lie brackets on V. Given $\mu \in \mathcal{L}_n$ and $U \in \mathrm{C}\big((0,1], \mathrm{GL}(V)\big)$, define the family of $\mu_\varepsilon \in \mathcal{L}_n$, $\varepsilon \in (0,1]$, by $\mu_\varepsilon(x,y) := U_\varepsilon{}^{-1}\mu(U_\varepsilon x, U_\varepsilon y)\ \forall\ x,y \in V$. If for any $x,y \in V$ there exists the limit $\lim_{\varepsilon\to+0}\mu_\varepsilon(x,y) =: \mu_0(x,y)$, then $\mathfrak{g}_0 = (V,\mu_0)$ is a well-defined Lie algebra and is called a *contraction* of the Lie algebra $\mathfrak{g} = (V,\mu)$. The procedure $\mathfrak{g} \to \mathfrak{g}_0$ providing $\mathfrak{g}_0$ from $\mathfrak{g}$ is also called a *contraction.* If a basis of V is fixed, the parameter matrix $U_\varepsilon = U(\varepsilon)$, $\varepsilon \in (0,1]$, is called the *contraction matrix* of the contraction $\mathfrak{g} \to \mathfrak{g}_0$.

Definition 2. The contraction $\mathfrak{g} \to \mathfrak{g}_0$ is called *a Inönü–Wigner (IW) contraction* if its matrix U_ε can be represented in the form $U_\varepsilon = AW_\varepsilon P$, where the matrices A and P are nonsingular and constant (i.e., they do not depend on ε) and $W_\varepsilon = \mathrm{diag}(\varepsilon^{\alpha_1},\dots,\varepsilon^{\alpha_n})$ for some $\alpha_1,\dots,\alpha_n \in \mathbb{R}$. The n-tuple of exponents $(\alpha_1,\dots,\alpha_n)$ is called the *signature* of the generalized IW-contraction $\mathfrak{g} \to \mathfrak{g}_0$. A *simple IW-contraction* is a generalized IW-contraction with signature consisting of zeros and ones.

The following assertion, which stood as a conjecture for a long time, was proved in [5].

Theorem 3. *Any generalized IW-contraction is equivalent to a generalized IW-contraction with an integer signature (and the same associated constant matrices).*

One of these conjectures was that any contraction of Lie algebras can be realized as a generalized IW-contraction. This is true for contractions between three-dimensional real or complex Lie algebras. Consider four-dimensional real Lie algebras defined, up to antisymmetry of Lie bracket, by the following nonzero commutation relations:

$$\begin{aligned}
2A_{2.1}\colon &\quad [e_1,e_2]=e_1,\ [e_3,e_4]=e_3;\\
A_1\oplus A_{3.2}\colon &\quad [e_2,e_4]=e_2,\ [e_3,e_4]=e_2+e_3;\\
A_{4.1}\colon &\quad [e_2,e_4]=e_1,\ [e_3,e_4]=e_2;\\
A_{4.10}\colon &\quad [e_1,e_3]=e_1,\ [e_2,e_3]=e_2,\ [e_1,e_4]=-e_2,\ [e_2,e_4]=e_1.
\end{aligned}$$

Hereafter we use the Mubarakzyanov's nomenclature for low-dimensional Lie algebras, and $\mathfrak{g}_{\dots}$ denotes the complexification of the algebra $A_{\dots}$. All contractions of four-dimensional real Lie algebras were realized in [1, 2] via generalized IW-contractions except two contractions, $2A_{2.1} \to A_1 \oplus A_{3.2}$ and $A_{4.10} \to A_1 \oplus A_{3.2}$. Since the complexifications of the algebras $2A_{2.1}$ and $A_{4.10}$ are isomorphic, there was only one exception for the complex case, $2\mathfrak{g}_{2.1} \to \mathfrak{g}_1 \oplus \mathfrak{g}_{3.2}$.

Theorem 4 ([4])**.** *(i) There exists a unique contraction between four-dimensional complex Lie algebras, $2\mathfrak{g}_{2.1} \to \mathfrak{g}_1 \oplus \mathfrak{g}_{3.2}$, that is not equivalent to a generalized IW-contraction.*

(ii) Precisely two contractions between four-dimensional real Lie algebras, $2A_{2.1} \to A_1 \oplus A_{3.2}$ and $A_{4.10} \to A_1 \oplus A_{3.2}$, cannot be realized as generalized IW-contractions.

Combining the results of [1, 2, 4] also yields the following assertion.

Theorem 5 ([4]). *Any generalized IW-contraction between four-dimensional complex (resp. real) Lie algebras is equivalent to one with parameter exponents in* $\{0,1,2,3\}$. *The exponents in* $\{0,1,2\}$ *suffice for all such contractions except* $2A_{2.1}\to A_{4.1}$, $A_{4.10}\to A_{4.1}$ *and* $\mathrm{so}(3)\oplus A_1\to A_{4.1}$ *in the real case and* $2\mathfrak{g}_{2.1}\to\mathfrak{g}_{4.1}$ *in the complex case, where the minimal tuple of exponents is* $(3,2,1,1)$.

Definition 6. The contraction $\mathfrak{g}\to\mathfrak{g}_0$ is called *diagonal* if its matrix U_ε can be represented in the form $U_\varepsilon=AW_\varepsilon P$, where A and P are constant nonsingular matrices and $W_\varepsilon=\mathrm{diag}(f_1(\varepsilon),\dots,f_n(\varepsilon))$ for some continuous functions $f_i\colon(0,1]\to\mathbb{F}\backslash\{0\}$.

Theorem 7 ([5]). *Any diagonal contraction is equivalent to a generalized IW-contraction with an integer signature.*

Consider the n-dimensional ($n\geqslant 5$) solvable real Lie algebras $\mathfrak{a}:=A_{5.38}\oplus(n-5)A_1$ and $\mathfrak{a}_0:=A_{2.1}\oplus A_{2.1}\oplus(n-4)A_1$ whose nonzero commutation relations are exhausted, up to antisymmetry of Lie bracket, by the following:

$$\mathfrak{a}\colon\quad [e_1,e_3]=e_3,\quad [e_2,e_4]=e_4,\quad [e_1,e_2]=e_5,\qquad \mathfrak{a}_0\colon\quad [e_1,e_3]=e_3,\quad [e_2,e_4]=e_4.$$

Theorem 8 ([3]). *The Euclidean norm of any contraction matrix that realizes the contraction of the algebra* $\mathfrak{a}$ *to the algebra* $\mathfrak{a}_0$ *approaches infinity at the limit point. The same is true for the complex counterpart of this contraction.*

Definition 9. A realization of a contraction with a matrix-function that is linear in the contraction parameter is called a *Saletan (linear) contraction*.

Theorem 10 ([6]). *Up replacing the algebras* $\mathfrak{g}$ *and* $\mathfrak{g}_0$ *with isomorphic ones, every Saletan contraction* $\mathfrak{g}\to\mathfrak{g}_0$ *is realized by a matrix of the canonical form*

$$E^{n_0}\oplus J_\varepsilon^{n_1}\oplus\dots\oplus J_\varepsilon^{n_s},\qquad \textit{or, equivalently,}\qquad E^{n_0}\oplus J_0^{n_1}\oplus\dots\oplus J_0^{n_s}+\varepsilon E^n,$$

where $n_0+\dots+n_s=n$, E^m *is the* $m\times m$ *identity matrix, and* J_λ^m *denotes the* $m\times m$ *Jordan block with an eigenvalue* λ.

Hence any Saletan contraction can be realized by a matrix of the form $AS_\varepsilon B$, where A and B are constant nonsingular matrices and the matrix-valued function S_ε is in the above canonical form. The tuple $(n_0;n_1,\dots,n_s)$, where $n_1,\ \dots,\ n_s$ constitute a partition of the dimension $n-n_0$ of the Fitting null component relative to U_0 and $n_0\in\{0,\dots,n\}$, is called the *signature* of this Saletan contraction.

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Weak Separation Condition coincides Finite Type Condition

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Geometric and Topological properties of self similar sets are well understood when the associated *Iterated Function System (IFS)* satisfies *Open Set Condition (OSC)*[1]. OSC fails if IFS is said to have overlaps. Lau and Ngai in [2] introduced *Weak separation property (WSP)* which allows limited overlaps of copies and is less restrictive than OSC. This separation property was extensively studied by Zerner [3]. The notion of separation allows to explicitly calculate the Hausdorff dimension of self-similar sets. But it can be still challenging. Another notion of separation termed as *Finite Type Condition (FTC)* introduced in [3] by , which enhances the domain of self similar sets. The question of equivalence was first raised in [4], they proved that it was not true in general for $d > 1$ IFS in $\mathbb{R}^n$. We characterise Weak separation property in terms of neighbourhood sets.

Notations and Definitions

Let $\mathcal{S} = \{S_1, ..., S_m\}$ be system of contracting maps on $\mathbb{R}$: $|S_i(x) - S_j(y)| = r_i|x - y|$, where $1 > r_i > 0$. Then $\exists$ unique compact set K such that $K = \bigcup_{i=1}^{m} S_i(K)$. K is called *self similar set* associated to IFS $\mathcal{S}$. OSC states that there is an open set $O \neq \phi$ such that $S_i(O) \subseteq O$ and $S_i(O) \cap S_j(O) = \phi$ for $i, j \in \{1, ..., m\}$. Let $I = \{1, ..., m\}$ be finite set of symbols and and $\mathbf{i}, \mathbf{j}$ be words from $I^* = \bigcup\{I^n : n = 1, 2, ...\}$. From [5] consider the following :

$$F = \{S_{\mathbf{i}}^{-1} \cdot S_{\mathbf{j}} \ ; \ \mathbf{i}, \mathbf{j} \in I^*\}$$

subset of topological group $\mathcal{G}$ of all similarities on $\mathbb{R}$. From [6] given any $a > 0$, let

$$I_a = \{\mathbf{i} = i_1 i_2 \dots i_n \in I^*; |r_{\mathbf{i}}| < a \leq |r_{i_1 i_2 \dots i_{n-1}}|\}$$

IFS satisfy weak separation property if there are $x \in \mathbb{R}$ and integer $l \in \mathbb{N}$ such that for any $a > 0$ and finite word σ, every closed ball with radius a, contains atmost l distinct elements of type $S_{\mathbf{i}}((S_\sigma(x))$ for $\mathbf{i} \in I_a$.

Definition : 1 - The notion of *Neighbourhood Sets* defined in [7] is very helpful to study finite type condition. For $\alpha \in \mathbb{Z}$, let $h_1, ..., h_{m_\alpha}$ be elements of set $\{S_{\mathbf{i}}(0), S_{\mathbf{i}}(1) : \mathbf{i} \in I_a\}$. Let $\mathcal{F}$ be union of all possible net intervals such that

$$\mathcal{F}_\alpha = \{[h_i, h_{i+1}] : 1 \leq i \leq m_\alpha\}$$

Suppose $\triangle \in \mathcal{F}$ and denote contraction map $T_\triangle(x) = rx + c$ where $r > 0$ such that $T_\triangle([0, 1]) = \triangle$. Similarity $T(x) = Lx + c$ is neighbourhood set of $\mathcal{F}_a$ if $\exists$ $\mathbf{i} \in I_a$ such that

$$\triangle \subseteq S_{\mathbf{i}}([0, 1]) \quad \text{and} \quad T = T_\triangle^{-1} \circ S_{\mathbf{i}}$$

Definition : 2 IFS satisfies *finite neighbourhood condition* if it has finite neighbourhood set.

The main result follows the following lemma.

Lemma : 1 Presume that finite neighbourhood condition holds for system $\mathcal{S}$. Then $\exists$ $l > 0$ such that any $1 \geq a > 0$, $\mathbf{i}, \mathbf{j} \in I_a$ and $p, q \in \{0, 1\}$ either

$$S_{\mathbf{i}}(p) = S_{\mathbf{j}}(q) \quad \text{or} \quad |S_{\mathbf{i}}(p) - S_{\mathbf{j}}(q)| \geq la$$

Lemma : 2 Suppose that $K = [0, 1]$ is self similar set of IFS $\mathcal{S}$ and WSP holds. For $\delta > 0$, $\exists$ a finite set $\mathcal{N}_\delta$ so that for any $a > 0$ and $\mathbf{i}, \mathbf{j} \in I_a$ either

$$\mu(S_{\mathbf{i}}([0, 1]) \cap S_{\mathbf{j}}([0, 1])) < \delta a \quad \text{or} \quad S_{\mathbf{i}}^{-1} \circ S_{\mathbf{j}} \in \mathcal{N}_\delta$$

Theorem : Let $K = [0,1]$ be the self similar set associated to system of contraction maps $\mathcal{S}$ such that weak separation property holds. Then finite neighbourhood condition is satisfied for $\mathcal{S}$.

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Topological data analysis for cosmology: theory and applications

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Cosmology and *Big data* or data analysis, of which topo-geometrical data analysis is rapidly becoming a main component, are both burgeoning and increasingly related fields at the moment. Cosmology is transitioning from a theoretical discipline towards one with increased focus on observations, resulting in a massive surge of data that demands increasingly more sophisticated methods to glean meaningful information. In a related development, geometry and topology have witnessed a tilt from purely theoretical fields towards strong focus on application. A foray into "big data" quickly brings to front two of the central statistical challenges of our times – detection and classification of structure in extremely large, high-dimensional, data sets. Among the most intriguing new approaches to this challenge is "TDA," or "topological data analysis," the primary aim of which is providing topologically informative pre-analyses of data, which serve as input to more quantitative analyses at a later stage. Algebraic and computational topology are the foundational pillars on which TDA rests.

I will present a survey on the theoretical and computational aspects of topological data analysis [1], simultaneously exploring up the application component via analyses of cosmological datasets. The dataset we will focus on is of the Cosmic Microwave Background, obtained by the recently concluded Planck mission, with a view to compare the observations with the predictions of the standard cosmological model, that predicts the initial conditions in the Universe to be that of an isotropic, homogeneous Gaussian random field [2]. At the epoch of recombination in the infant stage of the Universe, some 370,000 years after the Big Bang, matter and radiation separate for the first time, and radiation permeates freely in the Universe. This free-streaming radiation, that we observe as the Cosmic Microwave Background, encodes a treasure trove of information about the initial conditions and properties of matter distribution in the Universe [3].

The tentative outline of my presentation, in three parts, is as follows:

- A review of the theoretical background on geometry and topology consisting of Minkowski functionals, homology and its hierarchical extension persistent homology.
- A description of the main computational components for a variety of settings relevant to cosmological data sets, such as particle distributions and images in 2D and 3D. I will give a brief but in-depth account of the computational backbone, which relies on appropriate meshing of the domain, and hierarchical embedding of levelsets in filtration data structure.
- Building up on the first and the second item, I will present case studies involving the CMB and the SDSS dataset.

The above mentioned tentative structure is subject to time constraints, and the second item may be expunged for short duration of talk.

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Morse flows with singularities on boundary of 3-manifolds

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A flow X on a manifold with boundary ∂M is called *a Morse flow* if it satisfies the following conditions:

(1) the set of non-wandering points $\Omega(X)$ has a finite number of points that all are hyperbolic;
(2) if $u, v \in \Omega(X)$, then the unstable manifold $W^u(u)$ is transverse to the stable manifold $W^s(v)$ in $\mathrm{Int}M$;
(3) the restriction of X to ∂M is a Morse flow (the stable and unstable manifolds have a transversal intersection).

We consider Morse flows with singularities on ∂M. There are 6 types of singularities which are determined by their indices.

The pair (p, q) is called the index of a singular point where $p + q$ is equal to dimension of stable manifold of X and p is the dimension of the flow restricted to the boundary. In this case $p = 0, 1$ or 2 and $q = 0$ or 1. For example, a source has index (0,0) and a sink has index (2,1).

The surface F is the closure of intersection of Int M with boundary of a regular neighborhood for the union of the 1-dimensional stable manifolds.

Arcs and circles $\{u, U, v, V\}$ on F are intersections of unstable manifolds for singular points of index (1,0), (0,1),(1,1),(2,0) and the surface F.

The set (F, u, U, v, V) consisting of a surface with boundary, a set of circles and arcs embedded in it as described above is called a Morse flow diagram.

Theorem 1. *Two Morse-Smale flows on 3-manifold with a boundary are topologically trajectory equivalent if and only if their diagrams are homeomorphic.*

Morse flow diagrams have the following properties:

(1) $U_i, V_i \subset \mathrm{Int}M$, $\mathrm{Int}u_i, \mathrm{Int}v_i \subset \mathrm{Int}M$, $\partial u_i, \partial v_i \subset \partial M$;
(2) $\partial U_i \partial \cup_i u_i, \partial V_i \partial \cup_i v_i$;
(3) $U_i \cap U_j = \emptyset$ if $i \neq j$, $\quad u_i \cap u_j = \emptyset$ if $i \neq j$, $\quad V_i \cap V_j = \emptyset$ if $i \neq j$, $\quad v_i \cap v_j = \emptyset$ if $i \neq j$, $u_i \cap U_j = \emptyset$, $\quad v_i \cap V_j = \emptyset$, $\quad \partial u_i \cap \partial v_j = \emptyset$.
(4) U_k is a closed curve or it belongs to a left-hand turn cycle which consists of U_i and u_j; the similar property holds true for V_k.
(5) if we cut F along u_i and do spherical surgeries by U-cycles then we get a union of 2-disks.

Theorem 2. *If a surface F with 4 sets of curves has the properties 1-5, then it is a diagram of a Morse flow.*

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On the matrix equation $AX - YB = C$ over Bezout domains

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Let $\mathrm{R}_{m,n}$ be the set of $m \times n$ matrices over a commutative ring R with identity $e \neq 0$. Denote by I_n the identity $n \times n$ matrix and by $0_{m,n}$ the zero $m \times n$ matrix. For any matrix $A \in \mathrm{R}_{m,n}$ A^t denotes the transpose of A. We will denote by $GL(m,\mathrm{R})$ the set of invertible matrices in $\mathrm{R}_{m,m}$. We will write $C_{\downarrow i}$ for the the i-th column of the matrix $C \in \mathrm{R}_{m,n}$ and $\mathrm{vec}\,(C)$ will denote an ordered stock of columns of C, i.e.,

$$\mathrm{vec}\,(C) = \begin{bmatrix} C_{\downarrow 1} \\ C_{\downarrow 2} \\ \vdots \\ C_{\downarrow n} \end{bmatrix}.$$

In this note we present alternative methods for finding solutions of the Sylvester matrix equation

$$AX - YB = C, \tag{1}$$

where A, B and C are given matrices of suitable sizes over a commutative domain.

This equation has been considered by several authors including Roth [12] over a field, Hartwig [6] over a regular ring, Gustafson [5] over a commutative ring with identity, Emre and Silverman [3] over a polynomial ring, Özgüler [7] over a principal ideal domain, Dajić [2] over an associative ring with unit. In general, Gustafson [5] has proved that equation (1) over a commutative ring R with identity has a solution (X, Y) over R if and only if the matrices $\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$ and $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ are equivalent. This is a generalization of Roth's result [12], which gives the same criterion for the case, where R is a field. Similar considerations on solvability of equation (1) can be found in original paper [1].

1. Let R be a Bezout domain. Without reducing the generality we will assume that $A \in \mathrm{R}_{m,m}$, $B \in \mathrm{R}_{n,n}$ and $C \in \mathrm{R}_{m,n}$, and X, Y are unknown $m \times n$ matrices over R. Using the Kronecker product matrix equation (1) may be considered in the form of equivalent linear system (see [4])

$$(I_n \otimes A)\,\mathrm{vec}\,(X) - \left(B^t \otimes I_m\right)\mathrm{vec}\,(Y) = \mathrm{vec}\,(C).$$

Theorem 1. *Matrix equation (1) over Bezout domain R is solvable if and only if matrices*

$$\begin{bmatrix} (I_n \otimes A) & \left(B^t \otimes I_m\right) & 0_{mn,1} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} (I_n \otimes A) & \left(B^t \otimes I_m\right) & \mathrm{vec}\,(C) \end{bmatrix}$$

are column equivalent, i.e., the right Hermite normal forms of these matrices are the same.

Corollary 2. *Let $A_i \in \mathrm{R}_{m,m}$, $B_i \in \mathrm{R}_{n,n}$ and $C_i \in \mathrm{R}_{m,n}$, $i = 1, 2$. Matrix equations $A_1X - YB_1 = C_1$ and $A_2X - YB_2 = C_2$ have a common solution over Bezout domain R if and only if matrices*

$$\begin{bmatrix} (I_n \otimes A_1) & \left(B_1^t \otimes I_m\right) & 0_{mn,1} \\ (I_n \otimes A_2) & \left(B_2^t \otimes I_m\right) & 0_{mn,1} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} (I_n \otimes A_1) & \left(B_2^t \otimes I_m\right) & \mathrm{vec}\,(C_1) \\ (I_n \otimes A_2) & \left(B_2^t \otimes I_m\right) & \mathrm{vec}\,(C_2) \end{bmatrix}$$

are column equivalent, i.e., the right Hermite normal forms of these matrices are the same.

We were using results of papers [9] and [10] for proving Theorem 1.

2. In this parch R is a principal ideal domain. We denote by (a, b) the greatest common divisor of nonzero elements $a, b \in \mathrm{R}$. Let $A \in \mathrm{R}_{m,m}$ and $\mathrm{rank}\, A = r$. For the matrix A there exist matrices $U, V \in GL(m,\mathrm{R})$ such that $UAV = S_A = \mathrm{diag}(a_1, a_2, \ldots, a_r, 0, \ldots, 0)$ is the Smith normal form of A.

Theorem 3. *Let $A \in \mathrm{R}_{m,m}$, $B \in \mathrm{R}_{n,n}$, $C \in \mathrm{R}_{m,n}$ and* $\mathrm{rank} A = p$, $\mathrm{rank} B = q$. *Further, let $U_A, V_A \in GL(m, \mathrm{R})$ and $U_B, V_B \in GL(m, \mathrm{R})$ such that*

$$U_A A V_A = S_A = \mathrm{diag}(a_1, a_2, \dots, a_p, 0, \dots, 0), \quad U_B B V_B = S_B = \mathrm{diag}(b_1, b_2, \dots, b_q, 0, \dots, 0).$$

Matrix equation (1) is solvable over R *if and only if*

$$U_A C V_B = \left[\begin{array}{ccc|c} f_{11} & \dots & f_{1q} & \\ \vdots & \ddots & \vdots & 0_{p,n-q} \\ f_{p1} & \dots & f_{pq} & \\ \hline & 0_{m-p,q} & & 0_{m-p,n-q} \end{array}\right] \quad \text{and} \quad (a_i, b_j) | f_{ij} \,(\text{divides})$$

for all $i = 1, 2, \dots, p$ i $j = 1, 2, \dots, q$.

It is clear that if matrices $A \in \mathrm{R}_{m,m}$ and $B \in \mathrm{R}_{n,n}$ are nonsingular and $(\det A, \det B) = e$, then matrix equation (1) is solvable for an arbitrary matrix $C \in \mathrm{R}_{m,n}$.

Suppose that matrix equation (1) is solvable under the conditions of Theorem 3. Then for invariant factors a_i and b_j of matrices A and B respectively there exist $\alpha_{ij}, \beta_{ij} \in \mathrm{R}$ such that $a_i \alpha_{ij} - \beta_{ij} b_j = f_{ij}$ for all $i = 1, 2, \dots, p$ and $j = 1, 2, \dots, q$. Put

$$X_\alpha = \begin{bmatrix} \alpha_{11} & \dots & \alpha_{1q} \\ \vdots & \ddots & \vdots \\ \alpha_{p1} & \dots & \alpha_{pq} \end{bmatrix} \quad \text{and} \quad Y_\beta = \begin{bmatrix} \beta_{11} & \dots & \beta_{1q} \\ \vdots & \ddots & \vdots \\ \beta_{p1} & \dots & \beta_{pq} \end{bmatrix}.$$

Then for arbitrary matrices $P_{12}, Q_{12} \in \mathrm{R}_{p,n-q}$, $P_{21}, Q_{21} \in \mathrm{R}_{m-p,q}$ and $P_{22}, Q_{22} \in \mathrm{R}_{m-p,n-q}$ the pair of matrices

$$X_P = V_A^{-1} \begin{bmatrix} X_\alpha & P_{12} \\ P_{21} & P_{22} \end{bmatrix} V_B^{-1} \quad \text{and} \quad Y_Q = U_A^{-1} \begin{bmatrix} Y_\beta & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} U_B^{-1}$$

is the general solution of matrix equation (1). We note that Theorem 3 can be used for finding solutions with some properties of equation (1) (see [8] and [11]).

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On the metric equations generated by symplectic deformations on $P_2(C)$

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We analyzed in detail the cohomology structure of the symplectic form deformation and applied recently developed generalized transformations which were suggested in the classical works by Enneper and Weierstrass about one and half century ago and succeeded in reformulating the "symplectic" modification of the Monge-Ampere equation by means of specially constructed coordinates, related with the natural projector expansion on $T(P_2(\mathbb{C}))$ and found its special solutions. Let us consider a compact complex n-dimensional manifold M^n, endowed with the Kähler symplectic form $\omega \in \Lambda^2(M^n)$ and define the related Monge-Ampere equation, describing a deformation of this symplectic structure:

$$(\omega + i\partial\bar{\partial}\varphi)^n = (\exp f)\,\omega^n \tag{1}$$

under the normalizing conditions

$$\int_{M^n} (\exp f)\,\omega^n = \int_{M^n} \omega^n, \qquad \int_{M^n} \varphi\omega^n = 0, \tag{2}$$

where $\varphi \in C^\infty(M^n;\mathbb{R})$ is a real valued function on M^n and $\bar{\partial}$ is the complex ∂-bar differential, corresponding to the standard differential splitting $d = \partial \oplus \bar{\partial} : \Lambda(M^n) \to \Lambda(M^n)$ on the complex manifold M^n. In a general case it was supposed [16] that if the two-form $(\omega + i\partial\bar{\partial}\varphi) \in \Lambda^2(M^n)$ is real valued and the first Chern class $c_1(M^n) = 0$ of a Kähler manifold M^n, then there exists a Riemannian metric $g : T(M^n) \times T(M^n) \to \mathbb{R}$ of the Calabi-Yau type, whose holonomy group [5, 8] coincides with a subgroup of the Lie group SU(2), generating, in particular, a so called Einsteinian metric. The equation (1) is always [16] solvable, yet its holonomy groups, in general, not classified and its unitarity remains to be open.

We here also remark that there exists a slightly different modified Monge-Ampere type deformation equation

$$(\omega + dJ^*d\varphi)^n = (\exp f)\,\omega^n, \tag{3}$$

on a real symplectic manifold $\bar{M}^{2n} \simeq M^n$, where $f \in C^\infty(\bar{M}^{2n};\mathbb{R})$ and $J : T(\bar{M}^{2n}) \to T(\bar{M}^{2n}), J^2 = -I$, is a suitably chosen nonintegrable quasi-complex structure on the manifold $\bar{M}^{2n}$ and $J^* : T^*(\bar{M}^{2n}) \to T^*(\bar{M}^{2n})$ denotes its conjugate. It was proved [2] that if the structure $J : T(\bar{M}^{2n}) \to T(\bar{M}^{2n})$ is integrable, then the equation (3) reduces to the Monge-Ampere equation (1) on the related complex manifold $M^n \simeq \bar{M}^{2n}$ owing to the classical Newalander-Nirenberg [10] criterion. Otherwise, if the equation (3) is solvable for its arbitrarily chosen right hand side, then the quasi-complex structure $J : T(\bar{M}^{2n}) \to T(\bar{M}^{2n})$ proves to be necessary [2, 9, 11] a complex one, once more reducing the equation (3) to the Monge-Ampere equation (1).

In our note we are interested in the following "symplectic" modification

$$(\omega + dd^s\varphi)^2 = (\exp f)\,\omega^2 \tag{4}$$

of the Monge-Ampere (1) on the complex Kähler manifold $M^2 = P_2(\mathbb{C})$, where $\varphi \in \Lambda^2(M^2)$ is a searched for two-form and $d^s := (-1)^{k+1} \star_s d\star_s, dd^s = -d^s d,$ denotes the symplectic Hodge type differentiation. It is well known that any compact two-dimensional Kähler manifold M^2 with the Chern class $c_1(M^2) = 0$ is hiper-Kähler, possessing exactly three Kähler fundamental forms ω_I, ω_J and $\omega_K \in \Lambda^2(\bar{M}^4)$, corresponding to three complex structures I, J and $K : T(\bar{M}^4) \to T(\bar{M}^4)$. As for the compact projective two-dimensional Kähler manifold $M^2 = P_2(\mathbb{C})$ the Chern class $c_1(M^2) \neq 0$, it is not hiper-Kähler, its holomorphic volume two- form is not composed of the symplectic forms ω_J and $\omega_K \in \Lambda^2(\bar{M}^4)$. Notwithstanding this fact, based on the equalities (**??**) and the well known [1, 14, 15] relationship

$$\star_s \eta = -\eta \tag{5}$$

for an arbitrary "primitive" holomorphic volume two-form $\eta \in \Lambda^2_{hol}(M^2)$, satisfying the additional condition $\eta \wedge \omega = 0$, one easily derives that for any two cohomological "primitive" holomorphic volume two-forms Ω_1 and $\Omega_2 \in \Lambda^2_{hol}(M^2)$ there holds the following interesting relationship:

$$\Omega_1 - \Omega_2 = dd^s \psi \tag{6}$$

for some smooth two-from $\psi \in \Lambda^2(M^2)$, solving the problem (4) for the case when the symplectic structure $\omega \in \Lambda^2(M^2)$ is replaced by a holomorphic volume form $\Omega \in \Lambda^2_{hol}(M^2)$. Having analyzed in detail the cohomology structure of the two-form expression $(\omega + dd^s\varphi) \in \Lambda^2(M^2)$ and applied generalized transformations which were suggested in the classical works by Enneper [4] and Weierstrass [13] about one and half century ago and recently developed in [6], we succeeded in reformulating the "symplectic" modification of the Monge-Ampere (4) by means of specially constructed coordinates, related with the natural projector expansion from in $P_2(\mathbb{C})$ and find its special solutions.

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On a result of G. Pisier concerning Sidon sets

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Let G be a compact abelian group, Γ be its dual group, i.e., the set of all continuous characters on G. $C(G)$, $L_p(G), 0 < p \leq \infty$, are classical Banach spaces (integrals are considered with respect to the Haar measure on G). $M(G)$ denotes the Banach space of all regular Borel measures on G. Convolution operations are defined by the usual ways and are denoted by $\varphi \star f$, $\mu \star f$ for functions f, φ and a measure $\mu \in M(G)$. Below, H denotes a Hilbert space, $S_p(H)$, for $p \in (0, \infty)$, is the space of operators in H from Shatten-von Neumann class S_p (operators, whose singular numbers are in the classical space l_p). By S_∞ we denote the space of *all* operators in H. X, Y are Banach spaces, $L(X, Y)$ is a Banach space of all bounded linear operators from X to Y.

Definition 1. An operator $T : X \to Y$ *can be factored through an operator* from $S_p(H)$ (*through an* S_p*-operator*), if there are operators $A \in L(X, H)$, $U \in S_p(H)$ and $B \in L(H, Y)$ such that $T = BUA$. If T can be factored through an operator from $S_p(H)$, then we put $\gamma_{S_p}(T) = \inf ||A|| \, \sigma_p(U) \, ||B||$, where the infimum is taken over all possible factorizations of T through an operator from $S_p(H)$.

In [1], Giles Pisier gave a geometric characterization of Sidon subsets of Γ (for the definitions and formulations see [1, §4.b]). One of the main tool in his proof was the following result: For a function $f \in C(G)$ and a convolution operator $\star f : M(G) \to C(G)$, the necessary and sufficient condition for the set of Fourier coefficients $\hat{f} := \{\hat{f}(\gamma)\}$ to be absolutely summable is that the operator $\star f$ can be factored through a Hilbert space. It is clear that the last condition is the same as the condition "the operator $\star f$ can be factored through an S_∞-operator".

We present some generalizations of this result (proving simultaneously the above one). In particular, we have

Theorem 2. *Let* $f \in C(G)$, $0 < q \leq 1$ *and* $1/p = 1/q - 1$. *Consider a convolution operator* $\star f : M(G) \to C(G)$. *The set* $\hat{f}$ *of Fourier coefficients of* f *belongs to* l_q *if and only if the operator* $\star f$ *can be factored through a Schatten-von Neumann* S_p*-operator in a Hilbert space. Moreover, if* $\hat{f} \in l_q$, *then* $\gamma_{S_p}(\star f) = (\sum_{\gamma \in \Gamma} |\hat{f}(\gamma)|^q)^{1/q}$. *On the other hand,* $|| \star f|| = ||f||_{C(G)}$.

Instead of $M(G)$, we can consider the spaces $L_p(G)$ in the theorem (changing some values of parameters). Also, we can get some similar results for the factorizations of the convolution operators through the operators of the Lorentz-Schatten classes $S_{r,p}$ (associated with the Lorentz sequences spaces $l_{r,p}$).

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On the τ-placedness of space of the permutation degree

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We shall say that a set P is of the type G_τ in X if there exists a family $\gamma = \{U_\alpha : \alpha \in A, \mid A \mid \leq \tau\}$ of open sets in X such that $\bigcap_{\alpha\in A} U_\alpha = P$ (taken from [1]).

A subset $A \subset X$ is said to be τ-placed in X, if for each $x \in X \setminus A$ there exists a set $P \subset X$ of type G_τ in X such that $x \in P \subset X \setminus A$ (taken from [1]).

A permutation group X is the group of all permutations (i.s. one-one and onto mappings $X \to X$). A permutation group of a set X is usuallay denoted by $S(X)$. If $X = \{1, 2, ..., n\}$, then $S(X)$ is denoted by S_n, as well.

Let X^n be the n-th power of a compact X. The permutation group S_n of all permutations, acts on the n-th power X^n as permutation of coordinates. The set of all orbits of this action with quotient topology we denote by SP^nX. Thus, points of the space SP^nX are finite subsets (equivalence classes) of the product X^n. Thus two points $(x_1, x_2, ..., x_n), (y_1, y_2, ..., y_n) \in X^n$ are considered to be equivalent if there is a permutation $\sigma \in S_n$ such that $y_i = x_(\sigma(i))$ for all $i = 1, 2, ..., n$. The space SP^nX is called n-permutation degree of a space X. Equivalent relation by which we obtained space SP^nX is called the symmetric equivalence relation. The n-th permutation degree is always a quotient of X^n. Thus, the quotient map is denoted by as following: $\pi_n^s : X^n \to SP^nX$. Where for every $x = (x_1, x_2, ..., x_n) \in X^n$, $\pi_n^s((x_1, x_2, ..., x_n)) = [(x_1, x_2, ..., x_n)]$ is an orbit of the point $x = (x_1, x_2, ..., x_n) \in X^n$.

The concept of a permutation degree has generalizations. Let G be any subgroup of the group S_n. Then it also acts on X^n as group of permutations of coordinates. Consequently, it generates a G-symmetric equivalence relation on X^n. This quotient space of the product of X^n under the G-symmetric equivalence relation is called G-permutation degree of the space X and it is denoted by SP^n_GX. An operation SP^n_G is also the covariant functor in the category of compacts and it is said to be a functor of G-permutation degree. If $G = S_n$, then $SP^n_G = SP^n$. If the group G consists only of unique element, then $SP^n_GX = X^n$ (taken from [2]).

Theorem 1. *If the set SP^nA is τ-placed in SP^nX, then the set $(\pi_n^s)^{-1}(SP^nA)$ is also τ-placed in X^n.*

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The analogue of Darboux equation in Galilean space

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Let the surface F of the class $C^k(k \geq 2)$ in R_3^1 be given by the vector function $r = r(u,v)$ in the region $D \in R_2^1$, we will assume that the Cartesian coordinates R_3^1 are entered in x, y, z, and let $k$be the unit vector of the axis z. Each of the coordinates of the vector $r(u,v) = \{u,\ y(u,v),\ z(u,v)\}$ satisfies a certain differential equation. Let's deduce it, for example, for function $z(u,v)$.

Obviously, $z(u,v) = (r(u,v),\ k)$. Then

$$z_u = (r_u, k)\,,\ \ z_v = (z_v, k)\,,\ \ z_{uu} = (r_{uu}, k),\ \ z_{uv} = (r_{uv}, k),\ \ z_{vv} = (r_{vv}, k).$$

Using derivation formulas in R_3^1 we get

$$z_{uu} = \Gamma_{11}^2 z_v + L(n,k), \quad z_{uv} = \Gamma_{12}^2 z_v + M(n,k), \quad z_{vv} = \Gamma_{22}^2 z_v + N(n,k) \tag{1}$$

Where L, M, N is the coefficients of the second quadratic form, n is the surface normal.

If we introduce the notation

$$z_{11} = z_{uu} - \Gamma_{11}^2 z_v, \quad z_{12} = z_{uv} - \Gamma_{12}^2 z_v, \quad z_{22} = z_{vv} - \Gamma_{22}^2 z_v \tag{2}$$

then from (1) and (2) we obtain

$$z_{11} = L(n,k), \quad z_{12} = M(n,k), \quad z_{22} = N(n,k) \tag{3}$$

The unit normal vector is determined by the formula

$$n = \left\{0,\ \frac{z_v}{\sqrt{y_v^2 + z_v^2}},\ -\frac{y_v}{\sqrt{y_v^2 + z_v^2}}\right\}.$$

We have

$$(n,k) = -\frac{y_v}{\sqrt{y_v^2 + z_v^2}} \tag{4}$$

the for formula above gives a determination for the unit normal vector.

From (3) and (4) we obtain.

$$L = -\frac{z_{11}}{y_v}\sqrt{y_v^2 + z_v^2}, \quad M = -\frac{z_{12}}{y_v}\sqrt{y_v^2 + z_v^2}, \quad N = -\frac{z_{22}}{y_v}\sqrt{y_v^2 + z_v^2} \tag{5}$$

From equalities (5) and the formula for the Gaussian curvature $K = \frac{LN - M^2}{G}$ we obtain the Darboux equation $z_{11}z_{22} - z_{12}^2 = y_v^2 K$.

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On countable multiplicity of mappings

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A number of papers of mathematicians are devoted to the study of countable-to-one mappings, in particular M.M. Luzin, P.S. Alexandrov, A.M. Kolmogorov, B.O. Pasynkov, Yu.Yu. Trokhymchuk. In [1] a dense open set of points of local homeomorphism exists for any countable-to-one continuous mapping of two manifolds of equal dimensions was proved. Moreover, for the existence of a dense set of points of local homeomorphism, it suffices to require countable multiplicity of zero-dimensional mapping, even for points of some subset of the second category in the image [2]. In the one-dimensional case, the statement of the theorem remains valid for nowhere constant functions of the first Baire class with the Darboux property and with the set of countable levels of the second category in the image [3]. In paper [4] we consider the class of continuous on $[0, 1]$ functions preserving digit 1 in three-symbol Q_3–representation of a number and prove that any such function is countable-to-one and it has at most two infinite level sets. If we neglect some set of the first category, then with countable-to-one arbitrary B-measurable mapping of complete separable zero-dimensional uncountable space there exists a dense set of points of local homeomorphism [5]. It turns out that if quasi-continuous mapping of two complete separable metric spaces with the set of countable levels of the second category is nearly continuous on dense open set and is semi-open and pre-open, then it has a dense open set of points of local homeomorphism.

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Fuzzy ultrametrization of spaces of non-additive measures on fuzzy ultrametric spaces

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Recall that a triangular norm is a binary operation $*$ on the unit segment which is continuous, associative, commutative, monotone, for which 1 is the unit. The following are examples of triangular norms: min, $\cdot$ (multiplication), $a * b = \max\{a + b - 1, 0\}$ (Łukasiewicz t-norm).

Given a triangular norm $*$, we define a fuzzy metric on a set X as a function $M\colon X \times X \times (0, \infty) \to (0, 1]$ satisfying for all $x, y, z \in X$ and $s, t \in (0, \infty)$:

(1) $M(x, y, t) > 0$;
(2) $M(x, y, t) = 1$ if and only if $x = y$;
(3) $M(x, y, t) = M(y, x, t)$;
(4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;
(5) the function $M(x, y, \cdot)\colon (0, \infty) \to [0, 1]$ is continuous

(see, e.g., [2]).

A fuzzy metric is said to be a fuzzy ultrametric (a fuzzy non-Archimedean metric) if $* = \min$ and the following holds: (4') $M(x, y, t) * M(y, z, s) \leq M(x, z, \max\{t, s\})$. This is known to be equivalent to the following: (4") $M(x, y, t) * M(y, z, t) \leq M(x, z, t)$.

By $I(X)$ we denote the set of all idempotent measures on a compact Hausdorff space X (see [5]). This set is endowed with the weak* topology. In this way we obtain a functor in the category of compact Hausdorff spaces and continuous maps.

A standard construction allows us to consider the set of idempotent measures of compact support for any Tychonov space X; we keep the notation $I(X)$ for this set.

Let (X, M) be a fuzzy ultrametric space. A fuzzy ultrametric $\bar{M}$ on the set $I(X)$ is defined in [3]. The construction $(I(X), \bar{M})$ determines a functor in the category of fuzzy ultrametric spaces and non-expanding maps.

We continue the investigations of the mentioned paper as follows. The idempotent measure monad on the category of fuzzy ultrametric spaces is an idempotent counterpart of the probability measure monad on the same category which is introduced and investigated in [4]. Also, one can prove analogous results for the functor and monad of another class of non-additive measures, namely the max-min measures (see, e.g., [1]).

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About one class of continual approximate solutions with arbitrary density

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The kinetic Boltzmann equation is one of the central equations in classical mechanics of many-particle systems. For the model of hard spheres it has a form [1, 2]:

$$D(f) = Q(f, f). \tag{1}$$

We will consider the continual distribution [3]:

$$f = \int\limits_{\mathbb{R}^3} du \int\limits_0^{+\infty} d\rho \, \varphi(t, x, u, \rho) M(v, u, x, \rho), \tag{2}$$

which contains the local Maxwellian of special form describing the screw-shaped stationary equilibrium states of a gas (in short-screws or spirals). They have the form:

$$M(v, u, x, \rho) = \rho \left(\frac{\beta}{\pi}\right)^{\frac{3}{2}} e^{-\beta(v-u-[\omega\times x])^2}. \tag{3}$$

Physically, distribution (3) corresponds to the situation when the gas has an inverse temperature $\beta = \frac{1}{2T}$ and rotates in whole as a solid body with the angular velocity $\omega \in R^3$ around its axis on which the point $x_0 \in R^3$ lies,

$$x_0 = \frac{[\omega \times u]}{\omega^2}, \tag{4}$$

The square of this distance from the axis of rotation is

$$r^2 = \frac{1}{\omega^2}[\omega \times (x - x_0)]^2, \tag{5}$$

ρ is the arbitrary density, $u \in R^3$ is the arbitrary parameter (linear mass velocity for x), for which $x||\omega$, and $u + [\omega \times x]$ is the mass velocity in the arbitrary point x. The distribution (3) gives not only a rotation, but also a translational movement along the axis with the linear velocity

$$\frac{(\omega, u)}{\omega^2}\omega,$$

Thus, it really describes a spiral movement of the gas in general, moreover, this distribution is stationary (independent of t), but inhomogeneous.

The purpose is to find such a form of the function $\varphi(t, x, u, \rho)$ and such a behavior of all hydrodynamical parameters so that the uniform-integral remainder [3]

$$\Delta = \sup_{(t,x)\in\mathbb{R}^4} \int\limits_{\mathbb{R}^3} |D(f) - Q(f, f)| dv, \tag{6}$$

and its modification "with a weight":

$$\widetilde{\Delta} = \sup_{(t,x)\in\mathbb{R}^4} \frac{1}{1+|t|} \int\limits_{\mathbb{R}^3} |D(f) - Q(f, f)| dv, \tag{7}$$

become vanishingly small.

Also some sufficient conditions to minimization of remainder Δ and $\widetilde{\Delta}$ are found. In this work we succeeded a few to generalize results, which obtained in [3]. The obtained results are new and may be used with the study of evolution of screw and whirlwind streams.

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Painlevé VI Solutions From Equivariant ADHM Instanton Bundles

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We report on the paper [4] . Hitchin [3] had produced a pair of solutions $\lambda_0^{\pm}$ for the Painlevé VI differential equation from an $SL_2(\mathbb{C})$ action on the trivial bundle $E_0 \to P^3$ over complex projective space. We generalize to produce PVI solutions $\lambda_m^{\pm}$ for each nonnegative integer m from $SL_2(\mathbb{C})$ actions on the equivariant instanton bundles $E_m \to P^3$ constructed in [2] via an equivariant version of the Atiyah-Drinfeld-Hitchin-Manin construction [1].

Theorem 1. *For each nonnegative integer m, the equivariant instanton bundle E_m yields a pair of explicitly computable algebraic Painlevé VI solutions $\lambda_m^{\pm}(t)$, expressed implicitly in terms of the rational function*

$$t(w) = \frac{(1+w)\ (-3+w)^3}{(-1+w)\ (3+w)^3}$$

and a rational function of the form

$$\lambda_m^{\pm}(w) = \left(\frac{(-3+w)^2}{(-1+w)\ (3+w)}\right) \frac{(-1+w^2)\, f_m^{\pm}(w) + 8\, g_m^{\pm}(w)}{(3+w^2)\, f_m^{\pm}(w) - 24\, g_m^{\pm}(w)},$$

where $f_m^{\pm}$ and $g_m^{\pm}$ are even polynomials of degree at most $2m(m+1)$.

We have found explicit Okamoto transfromations $Q^{\pm 1}$ relating the two hierarchies of solutions $\lambda_m^{\pm}$ in a manner reminiscent of the familiar *creation operators* for eigenstates of the quantum harmonic oscillator. The following was proved case-by-case for a finite number of nonnegative integers m, and conjectured to hold for all nonnegative integers m:

Theorem 2. *For each nonnegative integer $m \leq 4$,*

$$\lambda_m^{+} = Q^m \lambda_0^{+}, \qquad \lambda_m^{-} = Q^{-m} \lambda_0^{-}.$$

We interpret each "creation operator" $Q^{\pm 1}$ as a "shadow" of a putative creation operator for equivariant instanton bundles E_m, which is indicated by the dashed arrows in the summary diagram:

$$\begin{tikzcd}
\lambda_0^+ \arrow[r, "Q"] & \lambda_1^+ \arrow[r, "Q"] & \lambda_2^+ \arrow[r, "Q"] & \lambda_3^+ \arrow[r, "Q"] & \lambda_4^+ \\
E_0 \arrow[u] \arrow[d] \arrow[r, dashed] & E_1 \arrow[u] \arrow[d] \arrow[r, dashed] & E_2 \arrow[u] \arrow[d] \arrow[r, dashed] & E_3 \arrow[u] \arrow[d] \arrow[r, dashed] & E_4 \arrow[u] \arrow[d] \\
\lambda_0^- \arrow[r, "Q^{-1}"] & \lambda_1^- \arrow[r, "Q^{-1}"] & \lambda_2^- \arrow[r, "Q^{-1}"] & \lambda_3^- \arrow[r, "Q^{-1}"] & \lambda_4^-
\end{tikzcd}.$$

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Asymptotic estimates for the widths of classes of periodic functions of high smothness

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Let L_p, $1 \le p \le \infty$, and C be the spaces of 2π–periodic functions with standart norms $\|\cdot\|_p$ and $\|\cdot\|_C$, respectively.

Denote by $C^{\psi}_{\bar{\beta},p}$, $1 \le p \le \infty$, the set of all 2π-periodic functions f, representable as convolution

$$f(x) = \frac{a_0}{2} + \frac{1}{\pi}\int\limits_{-\pi}^{\pi} \varphi(x-t)\Psi_{\bar{\beta}}(t)dt, \quad a_0 \in \mathbb{R}, \quad \varphi \in B^0_p = \{g \in L_p : \ \|g\|_p \le 1, \ g \perp 1\}, \tag{1}$$

with a fixed generated kernel $\Psi_{\bar{\beta}} \in L_{p'}$, $1/p + 1/p' = 1$, the Fourier series of which has the form

$$S[\Psi_{\bar{\beta}}](t) = \sum_{k=1}^{\infty} \psi(k)\cos\left(kt - \frac{\beta_k \pi}{2}\right), \quad \beta_k \in \mathbb{R}, \quad \psi(k) \ge 0. \tag{2}$$

A function f in the representation (1) is called $(\psi,\bar{\beta})$-integral of the function φ and is denoted by $\mathcal{J}^{\psi}_{\bar{\beta}}\varphi$ $(f = \mathcal{J}^{\psi}_{\bar{\beta}}\varphi)$. If $\psi(k) \ne 0$, $k \in \mathbb{N}$, then the function φ in the representation (1) is called $(\psi,\bar{\beta})$-derivative of the function f and is denoted by $f^{\psi}_{\bar{\beta}}$ $(\varphi = f^{\psi}_{\bar{\beta}})$. The concepts of $(\psi,\bar{\beta})$-integral and $(\psi,\bar{\beta})$-derivative was introduced by Stepanets (see [4]). Since $\varphi \in L_p$ and $\Psi_{\bar{\beta}} \in L_{p'}$, then the function f of the form (1) is a continuous function, i.e. $C^{\psi}_{\bar{\beta},p} \subset C$ (see [4, Proposition 3.9.2.]).

In the case $\beta_k \equiv \beta$, $\beta \in \mathbb{R}$, the classes $C^{\psi}_{\bar{\beta},p}$ are denoted by $C^{\psi}_{\beta,p}$. For $\psi(k) = k^{-r}, r > 0$, the classes $C^{\psi}_{\bar{\beta},p}$ and $C^{\psi}_{\beta,p}$ are denoted by $W^r_{\bar{\beta},p}$ and $W^r_{\beta,p}$, respectively. The classes $W^r_{\beta,p}$ are the well-known Weyl-Nagy classes. For $\psi(k) = e^{-\alpha k^r}$, $\alpha > 0$, $r > 0$, the classes $C^{\psi}_{\bar{\beta},p}$ and $C^{\psi}_{\beta,p}$ are denoted by $C^{\alpha,r}_{\bar{\beta},p}$ and $C^{\alpha,r}_{\beta,p}$, respectively. The sets $C^{\alpha,r}_{\beta,p}$ are well-known classes of the generalized Poisson integrals.

Let $\mathfrak{N}$ be some functional class from the space C ($\mathfrak{N} \subset C$). The quantity

$$E_n(\mathfrak{N})_C = \sup_{f\in\mathfrak{N}} E_n(f)_C = \sup_{f\in\mathfrak{N}} \inf_{T_{n-1}\in\mathcal{T}_{2n-1}} \|f - T_{n-1}\|_C \tag{3}$$

is called the best uniform approximation of the class $\mathfrak{N}$ by elements of the subspace $\mathcal{T}_{2n-1}$ of trigonometric polynomials T_{n-1} of the order $n-1$.

The order estimates for the best approximations $E_n(K)_C$ of classes $K = C^{\psi}_{\bar{\beta},p}$, $1 \le p \le \infty$, (and, hence, classes $W^r_{\beta,p}$, $C^{\alpha,r}_{\beta,p}$ and $C^{\psi}_{\beta,p}$) depending on rate of decreasing to zero of sequences $\psi(k)$ were obtained, in particular, in the works of Temlyakov (1993), Hrabova and Serdyuk (2013), Serdyuk and Stepanyuk (2014) etc.

If the sequences $\psi(k)$ decrease to zero faster than any geometric progression, then asymptotic equations of the best uniform approximations are even known (see [3] and the bibliography available there).

In [3] it was shown that for such classes $C^{\psi}_{\bar\beta,p}$ the following asymptotic equations take places

$$E_n(C^{\psi}_{\bar\beta,p})_C \sim \mathcal{E}_n(C^{\psi}_{\bar\beta,p})_C \sim \frac{\|\cos t\|_{p'}}{\pi}\psi(n), \quad 1\le p\le\infty, \quad \frac{1}{p}+\frac{1}{p'}=1, \tag{4}$$

where $\mathcal{E}_n(C^{\psi}_{\bar\beta,p}) = \sup\limits_{f\in C^{\psi}_{\bar\beta,p}} \|f-S_{n-1}(f)\|_C$, $S_{n-1}(f)$ is the partial Fourier sum of order $n-1$ of the function f, and $A(n){\sim}B(n)$ as $n\to\infty$ means that $\lim\limits_{n\to\infty} A(n)/B(n){=}1$.

For $p=\infty$ in the case of $K=W^r_{\bar\beta,\infty}, r>0$, and in the cases of $K=C^{\alpha,r}_{\bar\beta,\infty}, r\ge 1$, and $K=C^{\psi}_{\bar\beta,\infty}$ ($K=C^{\psi}_{\beta,\infty}$) for certain restrictions on sequences ψ and $\bar\beta$ the exact values of the best uniform approximations are known thanks to the works of Favard (1936, 1937), Akhiezer and Krein (1937), Krein (1938), Nagy (1938), Stechkin (1956), Dziadyk (1959, 1974), Sun (1961), Bushanskij (1978), Pinkus (1985), Serdyuk (1995, 1999, 2002) etc.

For $p=2$ and for arbitrary $\bar\beta=\beta_k\in\mathbb{R}, \sum\limits_{k=1}^{\infty}\psi^2(k)<\infty$ the exact values for the quantity $\mathcal{E}_n(C^{\psi}_{\bar\beta,2})_C$ are also known (see [2]).

Let K be a convex centrally symmetric subset of C and let $b_N(K,C), d_N(K,C), \lambda_N(K,C)$, and $\pi_N(K,C)$ be Bernstein, Kolmogorov, linear, and projection N-widths of the set K in the space C [1].

The results containing order estimates of the widths b_N, d_N, λ_N or π_N in the case of $K=C^{\psi}_{\bar\beta,p}$ (and, in particular, $W^r_{\beta,p}$ and $C^{\psi}_{\beta,p}$) can be found, for example, in the works of Tikhomirov (1976), Pinkus (1985), Kornejchuk (1987), Kashin (1977), Kushpel' (1989), Temlyakov (1990, 1993) etc.

Theorem 1. *Let $\{\beta_k\}_{k=1}^{\infty}$, $\beta_k\in\mathbb{R}$, and $\psi(k)>0$ satisfies the condition $\sum\limits_{k=1}^{\infty}\psi^2(k)<\infty$. Then for all $n\in\mathbb{N}$ the following inequalities hold*

$$\frac{1}{\sqrt{\pi}}\left(\frac{1}{\psi^2(n)}+2\sum_{k=1}^{n-1}\frac{1}{\psi^2(k)}\right)^{-\frac12} \le P_{2n}(C^{\psi}_{\bar\beta,2},C) \le P_{2n-1}(C^{\psi}_{\bar\beta,2},C) \le \frac{1}{\sqrt{\pi}}\left(\sum_{k=n}^{\infty}\psi^2(k)\right)^{\frac12}, \tag{5}$$

where P_N is any of the widths b_N, d_N, λ_N or π_N.

If, in adition, $\psi(k)$ satisfies the condition $\lim\limits_{n\to\infty}\max\left\{\psi(n)\left(\sum\limits_{k=1}^{n-1}\frac{1}{\psi^2(k)}\right)^{\frac12}, \frac{1}{\psi(n)}\left(\sum\limits_{k=n+1}^{\infty}\psi^2(k)\right)^{\frac12}\right\}=0$, then the following asymptotic equalities hold

$$\left.\begin{array}{l} P_{2n}(C^{\psi}_{\bar\beta,2},C) \\ P_{2n-1}(C^{\psi}_{\bar\beta,2},C)\end{array}\right\} = \psi(n)\left(\frac{1}{\sqrt{\pi}}+\mathcal{O}(1)\max\left\{\psi(n)\left(\sum_{k=1}^{n-1}\frac{1}{\psi^2(k)}\right)^{\frac12}, \frac{1}{\psi(n)}\left(\sum_{k=n+1}^{\infty}\psi^2(k)\right)^{\frac12}\right\}\right), \tag{6}$$

where $\mathcal{O}(1)$ are the quantities uniformly bounded in all parameters.

The equalities (6) are realized by trigonometric Fourier sums $S_{n-1}(f)$.

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Some generalizations of the known theorems of the type of geodesical unique definability

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The realized in [1] broadening to the noncompact but complete spaces of affine connection the well-known Hopf-Bochner-Uano techniques ([3], for example) on the grounding the so called vanishing theorems allowed to broad to the corresponding spaces some well-known theorems of the type of geodesical unique definability ([2], for example). In particular, it is grounded that the next theorems take place.

Theorem 1. *Complete connected noncompact Riemannian C^r-space V^n ($n > 2$, $r > 4$) with the positive defined metric tensor and the Einstein tensor that doesn't equal to zero identically, that satisfies the recurrence conditions*

$$T^{(\alpha\beta)}_{ijkl,mh} g^{mj} g^{hl} E^{ik}_{..} = T^{(\alpha\beta)}_{ijkl} W^{ijkl} + \frac{1}{n} T^{(\gamma j)}_{ijkl} R^{(\alpha|l|\beta)}_{\gamma\ .\ .} E^{ik}_{..} - \frac{1}{n} T^{(\alpha j)}_{ijkl} R^{\beta l}_{..} E^{ik}_{..} -$$
$$- \frac{1}{n} T^{(\beta j)}_{ijkl} R^{\alpha l}_{..} E^{ik}_{..} + T^{(\alpha\beta)}_{ijkl,m} W^{ijklm},$$

where

$$T^{\alpha\beta}_{ijkl} = n \left(\delta^{\beta}_{j} R^{\alpha}_{ikl} - \delta^{\beta}_{k} R^{\alpha}_{lji} \right) - g_{ik} \left(\delta^{\beta}_{j} R^{\alpha}_{.l} - R^{\alpha\ \ \beta}_{jl\ .} \right) + g_{jl} \left(\delta^{\beta}_{k} R^{\alpha}_{.i} - R^{\alpha\ \ \beta}_{ki\ .} \right),$$

"," means the corresponding covariant differentiation, doesn't admit non-trivial (different from the affine) geodesic mappings in the large.

Theorem 2. *Complete connected noncompact Riemannian C^r-space V^n ($n > 2$, $r > 4$) with the positive defined metric tensor and the Einstein tensor that doesn't equal to zero identically, that satisfies the recurrence conditions*

$$P^{(\alpha\beta)}_{ij,kh} g^{hi} E^{kj}_{..} = P^{(\alpha\beta)}_{ij,k} W^{ijk} + P^{(\alpha\beta)}_{ij} W^{ij},$$

where

$$P^{\alpha\beta}_{ij} = \delta^{\beta}_{i} R^{\alpha}_{.j} - \delta^{\beta}_{j} R^{\alpha}_{.i},$$

W^{ij} and W^{ijk} are some arbitrary tensors, correspondingly of the second and the third valence, doesn't admit non-trivial (different from the affine) geodesic mappings in the large.

Examples of the corresponding spaces are given.

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Some remarks on the Metrizability of F-metric spaces

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Abstract: In this talk, we will show that the newly introduced $\mathcal{F}$-metric space, introduced by Jleli and Samet in [1], is metrizable. Also, we deduce that the notions of convergence, Cauchy sequence, completeness due to Jleli and Samet for $\mathcal{F}$-metric spaces are equivalent to that of usual metric spaces. Moreover, we show that the Banach contraction principle in the context of $\mathcal{F}$-metric spaces is a direct consequence of its standard metric counterpart.

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The surfaces with the flat normal connection and the constant curvature of Grassmann image in Minkowski space

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The use of the concept of Grassmann image of the surface extends the circle of the problems, and it is the one of the methods of the study of the differential geometry of the surface. One the problems is the problem connected with the proof of the existence of the surface with given properties of its Grassmann image. In this paper, we consider the existence of the surface with the flat normal connection and constant curvature of its Grassmann image in Minkowski space $^1R_4$. The results of the solution of this problem depend on the type of Grassmann image. The concept of the normal connection of the submanifold of Riemannian manifold has been introduced by E. Cartan. The submanifolds with the flat normal connection have zero torsion. The important property of the surfaces with the flat normal connection is the existence of the parameterization, in which the first and two second quadratic forms can be reduced to the diagonal form simultaneously. The surfaces with the flat normal connection and their images in Minkowski space also have the additional properties.

If a time-like surface $V^2 \subset^1 R_4$ has the flat normal connection and the non-degenerate Grassman image, then that Grassman image is the time-like surface. In case of the space-like surface with a flat normal connection, its Grassmann image can be either a space-like surface or a time-like one.

The following existence theorems have been proved in this paper.

Theorem 1. *Let any $k \in [0,1]$ is given. Then in the space $^1R_4$ there exists the time-like C^3 class surface with the flat normal connection and the non-degenerate Grassmann image with the constant curvature $\bar{K} = k$. In the case $k = 0$, there is a surface with a constant Gauss curvature $K = 0$; if $k \in (0,1]$, then there exists the surface with the given function of the Gauss curvature $K = (\alpha_0^2 + 1)\beta(u^1)\delta(u^2)$, where $\alpha_0 = const, \beta(u^1), \delta(u^2))$ - the continuous functions.*

Theorem 2. *Let any $k \in (-\infty, -1]$ ($k \in (0, +\infty)$) is given. Then in the space $^1R_4$ there exists the space-like C^3 class surface with the flat normal connection and the non-degenerate space-like (time-like) Grassman image with constant curvature $\bar{K} = k$. If $k = 0$, then there exists the surface with the constant Gauss curvature $K = 0$; in the other cases there exists the surface with the given function of Gauss curvature $K(u^1, u^2) = (1 - \alpha_0^2)\beta(u^1)\delta(u^2)$, where $\alpha_0 = const, \alpha_0 \neq \pm 1$, $\beta(u^1), \delta(u^2)$ - the continuous functions.*

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The relation between T_0-topologies with the weight $2^{n-2} < k \leq 2^{n-1}$ on n-element set and T_0-topologies close to the discrete on $(n-1)$-element set

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It is common to speak that the topology on n-element set X has the weight k (or it belongs to k-class), if the topology contains k elements. Let us designate the minimum neighborhood of the element $a \in X$ by M_a. The concept of the vector of the topology (the nondecreasing sequence of the reduced by 1 powers of the minimum neighborhoods of all elements of X) have been introduced in [3]. In the work [3] the theorem on three types of the vectors of T_0-topologies with the weight $2^{n-1} < k \leq 2^n$ (close to the discrete topologies) has been proved:

1. $(0, ..., 0, \alpha_n)$, $1 \leq \alpha_n \leq n-1$;
2. $(\underbrace{0, ..., 0}_{k}, 1, ..., 1)$, $1 \leq k \leq n-2$, and $\bigcap_{m=k+1}^{n} M_m = \{y\}$;
3. $(0, ..., 0, 1, 1)$, $M_{n-1} \bigcap M_n = \emptyset$.

If T_0-topology on n-element set induces close to the discrete T_0-topology on some $(n-1)$-element set, then such topologies are called consistent.

The fact that T_0-topologies with the vectors $(0, ..., 0, \alpha_{n-1}, \alpha_n)$, $1 \leq \alpha_{n-1} \leq n-2$, $2 \leq \alpha_n \leq n-1$ (consistent with the close to discrete topologies of the first type) have weight $2^{n-2} < k \leq 2^{n-1}$ has been shown in [4]. The obtained results connected with the enumeration of T_0-topologies and the calculation of T_0-topologies in the individual classes have been compared with the results [1], [2].

T_0-topologies with the weight $2^{n-2} < k \leq 2^{n-1}$, which are consistent with the close to discrete topologies of the second and the third types have been considered in this paper. The fallowing facts have been proved: these topologies do not form new classes, and such topologies are contained in the same classes as T_0-topologies with vectors $(0, ..., 0, \alpha_{n-1}, \alpha_n)$.

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The Golomb and Kirch topologies on the set of nonzero integers

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The *Golomb* (resp. *Kirch*) topology on $\mathbb{Z}$ is generated by the subbase consisting of arithmetic progressions $a + b\mathbb{Z}$ where $a \in \mathbb{Z}$ and b is a (square-free) number, coprime with a. It is known that the Golomb (Kirch) topology on the subspace $\mathbb{Z}^\bullet = \mathbb{Z} \setminus \{0\}$ of non-zero integers is Hausdorff and (locally) connected. In the talk we shall discuss the homeomorphisms of the Golomb and Kirch topologies on $\mathbb{Z}^\bullet$ and $\mathbb{N}$.

Theorem 1 (Banakh, Spirito, Turek)**.** *The space $\mathbb{N}$ with the Golomb topology has a unique self-homeomorphism.*

Theorem 2 (Banakh, Stelmakh, Turek)**.** *The space $\mathbb{N}$ with the Kirch topology has a unique self-homeomorphism.*

Theorem 3 (Spirito)**.** *The space $\mathbb{Z}^\bullet$ with the Golomb topology has exactly two self-homeomorphisms.*

Theorem 4 (Stelmakh)**.** *The space $\mathbb{Z}^\bullet$ with the Kirch topology has exactly two self-homeomorphisms.*

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On symmetrization of univalent polynomials

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The problem of T-symmetrization of a univalent in the unit disc $\mathbb{D}$ function $f(z)$ is easy solvable by transformation $f^{(T)}(z) = \left[f(z^T)\right]^{1/T}$, $T = 1, 2, ...$ It does not work for univalent in $\mathbb{D}$ polynomials because the T-symmetrized function is not necessary a polynomial. We suggest a procedure which allows us to symmetrize several univalent in $\mathbb{D}$ polynomials, including Alexander polynomials, Brandt polynomials, de la Vallée Poussin polynomials, Fejér polynomials, Suffridge polynomials, and some others.

Projective invariants of linear planar 3-webs

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In this talk we present projective differential invariants of linear planar 3-webs. Linear 3-web on the plane $\mathbb{R}^2(x, y)$ is an unordered set of 3 linear foliations with the condition that leaves of any pair of foliations are transversal to each other. Any such web is defined by the set of 3 solutions $w = w(x, y)$ to the Euler equation (see [1])

$$w_y = ww_x.$$

We will consider actions of the group of projective transformations $SL_3(\mathbb{R})$ of the plane. This actions carries over to the space of solutions of the Euler equation. Representations of the Lie algebra $\mathfrak{sl}_3(\mathbb{R})$ by vector fields are

$$X_A = (a_{13} + (a_{11} - a_{33})x + a_{12}y - a_{32}xy - a_{33}x^2)\partial_x + \\ + (a_{23} + a_{21}x + (a_{22} - a_{33})y - a_{31}xy - a_{32}y^2)\partial_y,$$

where the matrix $A = \|a_{ij}\|_{i,j=1,2,3} \in \mathfrak{sl}_3(\mathbb{R})$.

Proposition 1. *The vector fields*

$$\overline{X}_A = X_A + \lambda_A(w)\partial_w,$$

where

$$\lambda_A(w) = (a_{21} - a_{31}y)w^2 + (a_{11} - a_{22} - a_{31}x + a_{32}y)w + a_{32}x - a_{12},$$

define representations of the Lie algebra $\mathfrak{sl}_3(\mathbb{R})$ *on the total space of the bundle*

$$\pi' : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2, \quad \pi'(x, y, w) \mapsto (x, y).$$

Moreover, the vector fields $\overline{X}_A$ *are symmetries of the Euler equation.*

Linear planar 3-webs are defined by a set of solutions w^1, w^2, w^3 to the Euler equation.

Proposition 2. *The vector fields*

$$\overline{X}_A = X_A + \lambda_A(w^1)\partial_{w^1} + \lambda_A(w^2)\partial_{w^2} + \lambda_A(w^3)\partial_{w^3}$$

define a representation of Lie algebra $\mathfrak{sl}_3(\mathbb{R})$ *on the total space of the bundle*

$$\pi : \mathbb{R}^2 \times \mathbb{R}^3 \to \mathbb{R}^2, \quad \pi(x, y, w^1, w^2, w^3) \mapsto (x, y).$$

Moreover, the vector fields $\overline{X}_A$ *are symmetries of the system of Euler equations*

$$w^1_y = ww^1_x, \quad w^2_y = w^2w^2_x, \quad w^3_y = w^3w^3_x. \tag{1}$$

System of equations (1) defines the submanifold

$$E_1 \subset J^1(\pi), \quad E_1 = \{w^1w^1_x - w^1_y = 0, w^2w^2_x - w^2_y = 0, w^3w^3_x - w^3_y = 0\},$$

where $J^1(\pi)$ is the bundle of 1-jets of sections of this bundle. Let $E_k \subset J^k(\pi)$ be a kth prolongation of this manifold.

A rational function I on the manifold E_k is called *a projective differential invariant* of linear 3-webs of order $\leq k$, if $\overline{X}^{(k)}_A(I) = 0$ on E_k for all $A \in \mathfrak{sl}_3(\mathbb{R})$. Here $\overline{X}^{(k)}_A$ is the kth prolongation of the vector field $\overline{X}_A$.

Solving the system of equations $\overline{X}^{(2)}_A(I) = 0$, we get the following result.

Theorem 3. *The field of rational projective differential invariants of order ≤ 2 of linear 3-webs is generated by invariants of order 2*

$$I_{21} = \frac{w^2_{xx}}{w^3_{xx}}, \quad I_{22} = -\frac{(-w^2 + w^3)w^1_x + (w^1 - w^3)w^2_x - w^3_x(w^1 - w^2)}{\sqrt{w^3_{xx}(w^2 - w^3)(w^1 - w^3)(w^1 - w^2)}}, \quad I_{23} = \frac{w^1_{xx}}{w^3_{xx}}.$$

This field separates regular $SL_3(\mathbb{R})$-orbits in E_2.

To describe the field of all projective differential invariants of linear 3-webs, we use the Lie-Tresse theorem (see [2]).

Theorem 4. *The field of rational projective differential invariants of linear 3-webs is generated by the basis invariants I_{21}, I_{22}, I_{23} and the invariant derivations*

$$\nabla_1 = -\frac{(-w^2 + w^3)w^1}{w^1w^2_x - w^1w^3_x - w^1_xw^2 + w^1_xw^3 + w^2w^3_x - w^2_xw^3}\frac{d}{dx}$$

$$+\frac{-w^2 + w^3}{w^1w^2_x - w^1w^3_x - w^1_xw^2 + w^1_xw^3 + w^2w^3_x - w^2_xw^3}\frac{d}{dy},$$

$$\nabla_2 = -\frac{(w^3 - w^1)w^2}{w^1w^2_x - w^1w^3_x - w^1_xw^2 + w^1_xw^3 + w^2w^3_x - w^2_xw^3}\frac{d}{dx}$$

$$+\frac{-w^2 + w^3}{w^1w^2_x - w^1w^3_x - w^1_xw^2 + w^1_xw^3 + w^2w^3_x - w^2_xw^3}\frac{d}{dy}.$$

This field separates regular orbits.

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Ultrametric spaces of $*$-measures

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Recall that an ultrametric on a set X is a metric d that satisfies the strong triangle inequality $d(x,y) \leq \max\{d(x,z), d(z,y)\}$, for all $x, y, z \in X$.

A triangular norm is a binary operation $*$ on the unit segment which is continuous, associative, commutative, monotone, for which 1 is the unit.

In [1], the functor M^* of $*$-measures acting on the category **Comp** of compact Hausdorff spaces is defined for any triangular norm $*$. A $*$-measures on a compact Hausdorff space X is a functional $\mu : C(X, [0,1]) \to [0,1]$ satisfying: 1) $\mu(c_X) = c, c \in [0,1]$, 2) $\mu(\varphi \vee \psi) = \mu(\varphi) \vee \mu(\psi)$, 3) $\mu(c * \varphi) = c * \mu(\varphi)$.

The set $M^*(X)$ of all $*$-measures on a compact Hausdorff space X is endowed with the weak* topology.

The space $M^*(X)$ of $*$-measures of compact support can be also considered for any Tychonov space X.

The aim of the talk is to consider the ultrametrization of the set $M^*(X)$ for any ultrametric space X. Given $r > 0$ we define the set $\mathcal{F}_r(X)$ of functions from $C(X, [0,1])$ constant on the balls of radius r.

Similarly as in [2] we define an ultrametric $\hat{d}$ on $M^*(X)$ by the formula $\hat{d}(\mu, \nu) = \inf\{r > 0 | \mu(\varphi) = \nu(\varphi)$ for all $\varphi \in \mathcal{F}_r(X)\}$.

We establish some topological and algebraic properties of the obtained ultrametric space $(M^*(X), \hat{d})$.

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Sweep of surfaces in Galilean space

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Galilean space R_3^1 is a three-dimensional affine space with a degenerate metric [1].

The basic geometric elements of a straight line, plane, and parallelism in Galilean space do not differ from these concepts of Euclidean space. Significantly different spatial motions of these spaces, that is, the transformation of space that preserves the distance between points.

Under the sweep of surface we mean a uniquely mapping of pieces of the surface at which the distance between the points and the angle between the lines are preserved. It is allowed to cut the surface into pieces and indicate the gluing methods [2].

B.M. Sultanov studied the sweep of surfaces consisting only of parabolic points [3]. These are cylinders and cones. It is shown that parabolic points of the surface are divided into two classes: parabolic and special parabolic.

It is proved that they have a different sweep on the plane. An example is given of cylinders equal in Euclidean space, but in the Galilean space one of them is a parabolic surface, the other is special parabolic. Moreover, they have different sweeps on the plane.

In this article, a surface sweep is obtained that is uniquely projected onto a general position plane in Galilean space.

Definition 1. If between the points of the surface $F \subset R_3^1$ and the points of the domain G in the plane Oxy, there is an unambiguous mapping, the distance between the corresponding points have the same order and equal, then the domain G - sweep is called a surface F in the plane Oxy.

In Euclidean space has a sweep only convex polyhedral cylindrical surface, cone. The degeneracy of the Galilean space metric allows for the unfolding of surfaces of a wider class.

Theorem 2. *The surface $F \in R_3^1$ - width $[a, b]$ and uniquely projected on the Oxy plane, has a sweep G on the band $a \leq x \leq b$ of the Oxy plane.*

Let D be a domain on the plane in general position Oxy, and $D = \{(x, y) \in R_2^1 : a \leq x \leq b; \varphi_1(x) \leq y \leq \varphi_2(x)\}$, where $\varphi_1(x), \varphi_2(x)$ are continuous functions in $[a, b]$.

Consider a surface $F : z = f(x, y) \quad (x, y) \in D$ with a boundary uniquely projecting onto the boundary of the domain D.

Theorem 3. *The surface $F : z = f(x, y)$ is deployed to the area $G = \{(x, y) \in R_2^1 : a \leq x \leq b; 0 \leq y \leq \int_{\varphi_1}^{\varphi_2} \sqrt{1 + f_y^2(x, y)} dy\}$ on the plane Oxy.*

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Maximal distance minimizers. Examples and properties

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I will talk about the sets which have the minimal length over the class of closed connected sets $\Sigma \subset \mathbb{R}^2$ satisfying the inequality $\max_{y \in M} \operatorname{dist}(y, \Sigma) \leq r$ for a given compact set $M \subset \mathbb{R}^2$ and some given $r > 0$.

For a given compact set $M \subset \mathbb{R}^2$ consider the functional

$$F_M(\Sigma) := \sup_{y \in M} \operatorname{dist}(y, \Sigma),$$

where Σ is a closed subset of $\mathbb{R}^2$ and $\operatorname{dist}(y, \Sigma)$ stands for the Euclidean distance between y and Σ. Consider the class of closed connected sets $\Sigma \subset \mathbb{R}^2$ satisfying $F_M(\Sigma) \leq r$ for some $r > 0$. We are interested in the sets of the minimal length (one-dimensional Hausdorff measure) $\mathcal{H}^1(\Sigma)$ over the mentioned class (*minimizers*).

It is known that for all $r > 0$ the set of minimizers is nonempty. It is proven also that for each minimizer of positive length the equality $F_M(\Sigma) = r$ holds. Furthermore the set of minimizers coincides with the set of solutions of the dual problem: minimize F_M over all closed connected sets $\Sigma \subset \mathbb{R}^2$ with prescribed bound on the total length $\mathcal{H}^1(\Sigma) \leq l$.

In [1] (for the plane) and in [2] (for the general case) some properties of minimizers have been proven:

(a) A minimizer cannot contain loops (homeomorphic images of circles).
(b) For every point $x \in \Sigma$ one of two statements is true:
  i there exists a point $y \in M$ (may be not unique) such that $\operatorname{dist}(x, y) = r$ and $B_r(y) \cap \Sigma = \emptyset$;
  ii there exists an $\varepsilon > 0$ such that $S_\Sigma \cap B_\varepsilon(x)$ is either a segment or a regular tripod, i.e. the union of three segments with an endpoint in x and relative angles of $2\pi/3$.

The minimizers for some sets M are known (see pictures) although usually this is not an easy task. Recently (see [3]) at the plane the regularity of minimizers was proved.

Theorem 1. *Let Σ be a maximal distance minimizer for a compact set $M \subset \mathbb{R}^2$. Then*

(i) *Σ is a union of a finite number of arcs (injective images of the segment $[0; 1]$).*
(ii) *The angle between each pair of tangent rays at every point of Σ is greater or equal to $2\pi/3$. The number of tangent rays at every point of Σ is not greater than 3. If it is equal to 3, then there exists such a neighbourhood of x that the arcs in it coincide with line segments.*

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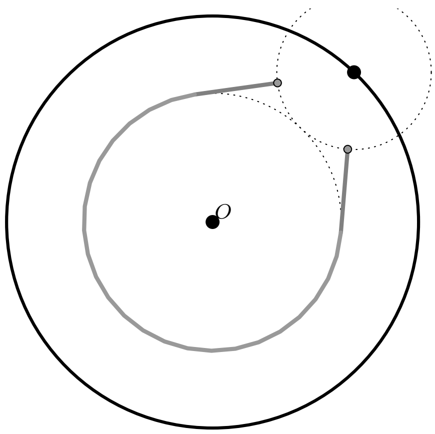

FIGURE 1.1. An example where $M = \partial B_R(O)$, where $R > 4.98r$.

FIGURE 1.2. An example where $M := \partial B_r([AG])$, $\Sigma = [AG]$.

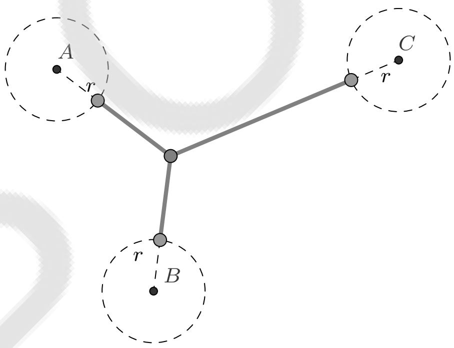

FIGURE 1.3. An example where $M := \{A, B, C\}$, Σ is a tripod.

Analog of Menchov-Trokhimchuk theorem for monogenic functions in three-dimensional commutative algebra

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Many scientists worked on finding new weaker conditions for holomorphicity of complex-valued functions: H. Bohr, H. Rademacher, D. Menchov [1], V. Fedorov, G. Tolstov, Y. Trokhimchuk [2, 3], G. Sindalovski, D.Teliakovski, E. Dolzhenko, M. Brodovich and their multidimencional generalizations: A. Bondar, V. Siryk, O. Gretskii.

Here is one of Menchov conditions: *function $F(\xi)$ satisfies K''' condition in point ξ_0 if exists limit*

$$\lim_{\xi\to\xi_0}\frac{F(\xi)-F(\xi_0)}{\xi-\xi_0}, \tag{1}$$

where ξ belongs to union of two noncollinear rays with common starting point ξ_0.

D. Menchov [1] has prowed that fulfillment of the condition K''' in any point of domain D (excluding not more than countable set) is sufficient for conformity of mapping F in case if $F: D\to\mathbb{C}$ is continuous univalent function. Y. Trokhimchuk [2] has removed univalency condition in following theorem.

Menchov-Trokhimchuk Theorem. *If function $F: D\to\mathbb{C}$ is continuous in domain D and in every its point, excluding not more than countable set, condition K''' is fulfilled, then function F is holomorphic in domain D.*

Analog of Menchov-Trokhimchuk Theorem for monogenic functions in space E_3.

Let $\mathbb{A}_3$ be 3-dimencional commutative associative algebra with unit 1 over the field $\mathbb{C}$ with basis $\{1,\rho,\rho^2\}$, such that $\rho^3=0$.

Let fix the real 3-dimencional subspace $E_3:=\{\zeta=xe_1+ye_2+ze_3 : x,y,z\in\mathbb{R}\}\subset\mathbb{A}_3$, where the vectors e_1,e_2,e_3 — are linearly independent over the real field $\mathbb{R}$, but, in general, not a basis of the algebra $\mathbb{A}_3$. Only one condition should be fulfilled: image of the E_3 under the mapping f is whole complex plane (see [5, 6]).

Function $\Phi'_G\colon\Omega\longrightarrow\mathbb{A}_3$ is called Gâteaux derivative of function $\Phi\colon\Omega\longrightarrow\mathbb{A}_3$, with domain $\Omega\subset E_3$, if in any point $\zeta\in\Omega$ exists element $\Phi'_G(\zeta)\in\mathbb{A}_3$ such that

$$\lim_{\delta\to 0+0}\left(\Phi(\zeta+\delta h)-\Phi(\zeta)\right)\delta^{-1}=h\Phi'_G(\zeta)\quad\forall h\in E_3\,. \tag{2}$$

Function $\Phi\colon\Omega\longrightarrow\mathbb{A}_3$ is called *monogenic* in domain $\Omega\subset E_3$, if Φ is continuous and has Gâteaux derivative in any point of Ω (see [8, 11]).

Intersection of radical of algebra $\mathbb{A}_3$ with linear space E_3 is the set of non-invertible elements which belongs to E_3. This set is the straight line $L:=\{c\,l : c\in\mathbb{R}\}$, with direction vector $l\in E_3$. Preimage of any point $\xi\in\mathbb{C}$ in E_3 under the mapping f is the straight line $L^\zeta:=\{\zeta+c\,l : c\in\mathbb{R}\}$, where ζ — element from E_3 such that $\xi=f(\zeta)$. Obviously, line L^ζ is parallel to line L.

Let domain $\Omega\subset E_3$ is convex in direction of straight line L (domain is called *convex in direction of straight line*, if it contains every segment joining two points of domain and parallel to this straight line).

Consider next hypercomplex analog of Menshov condition K''' in algebra $\mathbb{A}_3$ for functions $\Phi : \Omega \to \mathbb{A}_3$, defined in domain $\Omega \subset E_3$.

Definition 1. Let say, that function $\Phi : \Omega \to \mathbb{A}_3$ is fulfilled condition $K'''_{\mathbb{A}_3,E_3}$ at point $\zeta \in \Omega$, if exists element $\Phi_*(\zeta) \in \mathbb{A}_3$ such that equation

$$\lim_{\delta\to 0+0} \left(\Phi(\zeta+\delta h) - \Phi(\zeta)\right)\delta^{-1} = h\Phi_*(\zeta) \tag{3}$$

is fulfilled for three vectors h: h_1, h_2 and $h_3 = l$ or $h_3 = -l$, which are the basis of space E_3.

Theorem 2. *Let the domain $\Omega \subset E_3$ is convex in direction of straight line L, function $\Phi : \Omega \to \mathbb{A}_3$ is continuous in Ω and fulfill condition$K'''_{\mathbb{A}_3,E_3}$ in all points $\zeta \in \Omega$, except not more than countable set. Then:*

1) function Φ is monogenic in domain Ω;

2) function Φ extends to function monogenic in domain Π. This extension is unique and given by equality

$$\Phi(\zeta) = \frac{1}{2\pi i}\int\limits_{\gamma} \Big(F_0(\xi) + F_1(\xi)\rho + F_2(\xi)\,\rho^2\Big)(\xi-\zeta)^{-1}\,d\xi\,, \tag{4}$$

for all $\zeta \in \Pi$;

3) monogenic extension (4) of function Φ is differentiable in the sense of Lorch [10] in Π.

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Onto the some dynamic applications via quaternions

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Quaternion algebra, which generate the complex numbers in four dimensions, is one of the significant tools for not only mathematics but also physical applications [1, 2]. Quaternionic representations can be used some subfields of physics such as classical mechanics [3, 4], quantum mechanics [5], electromagnetism [6], linear gravity [7, 8], plasma [9] and fluid systems [10] etc. In this work, the quaternions with real coefficients and their some properties have been defined. By this way, the quaternionic descriptions of the rotation, translation and both two motions of the rigid body have been written in a detail manner and the applicable examples have been given. Then, the force and torque terms on the object have been presented and exemplified by using quaternion concept. Moreover, the manual operations have also been verified with the help of computer programs such as Mathematica and Matlab [11, 12]. As a consequence, it is said that the quaternion algebra is the important and practical mathematical structures for applicable sciences.

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On dynamical systems with a prescribed globally bp-attracting set

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Given an arbitrary fixed nonempty closed subset $\mathcal{C} \subset \mathbb{R}^n$, we propose an explicit method to construct a dynamical system which admits the regular part of $\mathcal{C}$ as globally bp-attracting set, i.e. a closed and invariant set which attracts every bounded positive orbit of the dynamical system. We apply this result in order to provide an explicit method of leafwise asymptotic bp-stabilization of the regular part of an a-priori given invariant set of a conservative system. The theoretical results are illustrated for the completely integrable case of the Rössler dynamical system.

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On certain fractal-based estimations of subsidence volume

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In [1, 2], the particle size distribution $N_s(L > d_s)$ was defined as the number of particles being of any size L larger than d_s, where d_s runs over the real numbers. In the same way we can introduce the particle size distribution by volume $V_s(L > d_s)$ (and by mass $M_s(L > d_s)$) as the volume (mass) of particles being of any size L larger than d_s, where d_s runs over the real numbers. Certainly, $N_s(L > d_s)$, $V_s(L > d_s)$ and $M_s(L > d_s)$ are real functions. The particle size distribution $N_s(L > d_s)$ has fractal dimension D_s if

$$N_s(L > d_s) = \gamma {d_s}^{-D_s},$$

where γ is a constant coefficient.

Under some additional conditions of fractal nature of the loess soil and developing methods introduced in [3, 4, 5] we obtained certain predictive estimations of the coefficient of porosity after the disintegration of micro-aggregates. In this note we obtain some estimations of soil subsidence volume, based on the introduced above fractal dimension.

The particles forming the ground may have only a finite set of sizes. We denote these sizes $d_1, d_2, ..., d_{n-1}, d_n$ ranging in decreasing order from the largest. We assume that $\alpha = \alpha_j = d_j/d_{j-1}$, where $2 \leqslant j \leqslant n$, does not depend on j. This assumption corresponds to the idea of the self-similarity of fractal structures. In addition, all known mathematical fractals are constructed on this principle. As the structures formed by particles of a fixed size are self-similar, we also assume that all these structures have the same coefficient of porosity k_p as well as the same porosity $K_p = k_p/(1 + k_p)$. We discovered that under such conditions two different situations may occurred. Let k' be the coefficient of porosity and K' be the porosity of the soil after the disintegration of micro-aggregates.

Theorem 1. *In the above denotations we have :*

1. *if* $K_p \geq \alpha^{3-D_s}$ *then* $k' = \frac{(1+k_p)(\alpha^{3-D_s}-1)}{(\alpha^{3-D_s})^n-1} - 1$ *and* $K' = 1 - \frac{(\alpha^{3-D_s})^n-1}{(1+k_p)(\alpha^{3-D_s}-1)}$ *;*
2. *if* $K_p < \alpha^{3-D_s}$ *then* $k' = \frac{k_p(1-\alpha^{3-D_s})}{1-(\alpha^{3-D_s})^n}$ *(5.18) and* $K' = \frac{k_p(1-\alpha^{3-D_s})}{k_p(1-\alpha^{3-D_s})+1-(\alpha^{3-D_s})^n}$ *.*

The details of our experiments and techniques are described in [4].

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Modular knots obey the Chebotarev law

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A *Chebotarev link* in S^3 is an analogue of the set of all prime numbers in $\mathbb{Z}$. It would play an important roll in arithmetic topology, especially when we formulate an analogue of the idelic class field theory for 3-manifolds [Uek21a] (see also [Mor12, Nii14, NU19, Mih19]). Here is the definition:

Definition 1 (The Chebotarev law). Let $(K_i)_i = (K_i)_{i \in \mathbb{N}_{>0}}$ be a sequence of disjoint knots in a 3-manifold M. For each $n \in \mathbb{N}_{>0}$ and $j > n$, put $L_n = \cup_{i \leq n} K_i$ and let $[K_j]$ denote the conjugacy class of K_j in $\pi_1(M - L_n)$. We say that $(K_i)_i$ *obeys the Chebotarev law* if the density equality

$$\lim_{\nu \to \infty} \frac{\#\{n < j \leq \nu \mid \rho([K_j]) = C\}}{\nu} = \frac{\#C}{\#G}$$

holds for any $n \in \mathbb{N}_{>0}$, any surjective homomorphism $\rho : \pi_1(M - L_n) \to G$ to any finite group, and any conjugacy class $C \subset G$.

In order to answer Mazur's question on the existence of such a link in S^3 [Maz12], by using Parry–Pollicott's zeta functions of symbolic dynamics [PP90], McMullen proved a highly interesting theorem:

Proposition 2 ([McM13, Theorem 1.2]). *Let $(K_i)_i$ be the closed orbits of any topologically mixing pseudo-Anosov flow on a closed 3-manifold M, ordered by length in a generic metric. Then $(K_i)_i$ obeys the Chebotarev law.*

Applying his theorem to the monodromy suspension flow of the figure-eight knot K and noting that the Chebotarev law persists under Dehn surgeries, he constructed a Chebotarev link containing K in S^3 [McM13, Corollary 1.3]. We refine his construction in two ways to verify the following assertion:

Theorem 3 ([Uek21b, Theorem 3]). *Let L be a fibered hyperbolic link in S^3 and let $(K_i)_i$ denote the sequence of knots consisting of the closed orbits of the suspension flow of the monodromy map and L itself. Then $(K_i)_i$ obeys the Chebotarev law, if ordered by length with respect to a generic metric.*

The union $\mathcal{L} = \cup_i K_i$ is a stably Chebotarev link*, that is, for any finite branched cover $h : M \to S^3$ branched along any finite link in $\mathcal{L}$, the inverse image $h^{-1}(\mathcal{L})$ is again Chebotarev.*

One way is to extend McMullen's theorem for *generalized pseudo-Anosov flows*, which allow 1-pronged singular orbits. The other is to invoke the notion of *rational Fried surgeries*, which produce many (generalized) pseudo-Anosov flows.

Our refinement further provides a new example called modular knots, that are also known as Lorenz knots. Let $\mathbb{H}^2 = \{z \in \mathbb{C} \mid \mathrm{Im} z > 0\}$ denote the upper half plane. The unit tangent bundle of the modular orbifold $\mathrm{PSL}_2\mathbb{Z} \setminus \mathbb{H}^2$ is well-known to be homeomorphic to both the quotient space $\mathrm{PSL}_2\mathbb{Z} \setminus \mathrm{PSL}_2\mathbb{R} \cong \mathrm{SL}_2\mathbb{Z} \setminus \mathrm{SL}_2\mathbb{R}$ and the exterior of a trefoil K in S^3. A flow on $\mathrm{PSL}_2\mathbb{Z} \setminus \mathrm{PSL}_2\mathbb{R}$ historically called *the geodesic flow* is defined by multiplying $\left(\begin{smallmatrix} e^t & 0 \\ 0 & e^{-t} \end{smallmatrix}\right)$ on the right, and its closed orbits are called *modular knots*. For each primitive hyperbolic element γ in $\mathrm{SL}_2\mathbb{Z}$, we may define the corresponding modular knot C_γ by $C_\gamma(t) = M_\gamma \left(\begin{smallmatrix} e^t & 0 \\ 0 & e^{-t} \end{smallmatrix}\right)$ $(0 \leq 0 \leq \log \xi_\gamma)$, where $M_\gamma^{-1} \gamma M_\gamma = \left(\begin{smallmatrix} \xi_\gamma & 0 \\ 0 & \xi_\gamma^{-1} \end{smallmatrix}\right)$ with $\xi_\gamma > 1$. Every modular knot admits such a presentation. By virtue of Bonatti–Pinsky's nice compactification [BP20], we obtain the following:

Theorem 4 ([Uek21b, Theorem 4]). *Modular knots and the missing trefoil in S^3 obey the Chebotarev law, if ordered by length in a generic metric.*

As a corollary, we obtain a result on a function with arithmetic origin. The discriminant function $\Delta(z) = q\prod_{n=1}^{\infty}(1-q^n)^{24}$ with $q = e^{2\pi\sqrt{-1}z}$, $z \in \mathbb{H}^2$ is a well-known modular function of weight 12. *The Dedekind symbol* Φ and *the Rademacher symbol* Ψ are the functions $\mathrm{SL}_2\mathbb{Z} \to \mathbb{Z}$ satisfying

$$\log\Delta(\gamma z) - \log\Delta(z) = \begin{cases} 6\log(-(cz+d)^2) + 2\pi i\Phi(\gamma) & \text{if } c \neq 0, \\ 2\pi i\Phi(\gamma) & \text{if } c = 0, \end{cases}$$

$$\Psi(\gamma) = \Phi(\gamma) - 3\mathrm{sgn}(c(a+d))$$

for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2\mathbb{Z}$ acting on $z \in \mathbb{C}$ via the Möbius transformation $\gamma z = \frac{az+b}{cz+d}$. Here we take a branch of the logarithm so that $-\pi \leq \mathrm{Im}\log z < \pi$ holds. This Ψ factors through the conjugacy classes of $\mathrm{PSL}_2\mathbb{Z}$ and satisfies $\Psi(\gamma^{-1}) = -\Psi(\gamma)$ for any γ.

The Rademacher symbol Ψ is known to be a highly ubiquitous function. Indeed, Atiyah proved the equivalence of seven definitions rising from very distinct contexts [Ati87], whereas Ghys gave further characterizations ([BG92], [Ghy07, Sections 3.3–3.5], [DIT17, Appendix]), proving that *for each primitive hyperbolic $\gamma \in \mathrm{SL}_2\mathbb{Z}$, the linking number between the modular knot C_γ and the missing trefoil K coincides with the Rademacher symbol*, namely,

$$\mathrm{lk}(C_\gamma, K) = \Psi(\gamma)$$

holds. Theorem 4 for $\rho(\gamma) = \mathrm{lk}(C_\gamma, K)$ mod m together with some arguments yield the following.

Corollary 5 ([Uek21b, Corollary 9]). *Suppose that γ runs through primitive hyperbolic elements of $\mathrm{SL}_2\mathbb{Z}$. For any $m \in \mathbb{Z}_{>0}$ and $k \in \mathbb{Z}/m\mathbb{Z}$, we have*

$$\lim_{\nu\to\infty} \frac{\#\{\gamma \mid |\mathrm{tr}\gamma| < \nu,\ \Psi(\gamma) = k \text{ in } \mathbb{Z}/m\mathbb{Z}\}}{\#\{\gamma \mid |\mathrm{tr}\gamma| < \nu\}} = \frac{1}{m}.$$

The similar arguments may be applicable to other Fuchsian groups. Modular knots for triangle groups around any torus knot in S^3 will be finely studied in [MU21].

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The Terence Tao set and the Collatz conjecture

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The Collatz conjecture is an open problem in number theory stablished in 1937 by Lothar Collatz and can be stated as follows: If $f : \mathbb{N} \to \mathbb{N}$ is the function define by:

$$f(n) = \begin{cases} \frac{n}{2} & ; n \text{ } is \text{ } even \\ 3n+1 & ; n \text{ } is \text{ } odd \end{cases}$$

the conjecture says that given $n \in \mathbb{N}$, there exists $k > 0$ such that $f^{(k)}(n) = 1$ and the only orbit is $\{1, 2, 4\}$

In 2019, Terence Tao showed, in the context of the Collatz conjecture, that almost all $n \in \mathbb{N}$ belong to the set $W = \{n \in \mathbb{N} : min(O(n)) < f(n)\}$. In this paper we prove that the Collatz conjecture is true if and only if the set W is connected in $\mathbb{N}$ with the primal topology τ_f, where τ_f is the topology on $\mathbb{N}$ given by the open sets as those subset θ of $\mathbb{N}$ such that $f^{-1}(\theta) \subset \theta$.

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Conformal mappings in Hardy-type spaces

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Let $H^p(\mathbb{C}_+), 1 \le p < +\infty$, [1] be the Hardy space of analytic in the half-plane $\mathbb{C}_+ = \{z : \Re z > 0\}$ functions, for which

$$\| f \| = \sup_{x>0} \left\{ \int_{-\infty}^{+\infty} |f(x+iy)|^p dy \right\}^{1/p} < +\infty.$$

Let $D_\sigma = \{z : \Re z < 0, |\Im z| < \sigma\}$, $D_\sigma^* = \mathbb{C} \setminus \overline{D}_\sigma, \sigma > 0$.

Definition 1. Let $E^p(D_\sigma)$ and $E^p(D_\sigma^*)$, $1 \le p < +\infty, \sigma > 0$, be the spaces of analytic functions in the domains D_σ and D_σ^* respectively, for which

$$\sup \left\{ \int_\gamma |f(z)|^p |dz| \right\}^{1/p} < +\infty,$$

where supremum is taken over all segments γ, that are contained in D_σ and D_σ^* respectively.

We consider the properties of functions in the half-strip D_σ and in the exterior of half-strip D_σ^*. In [2] considered spaces $E^p(D_\sigma)$ and $E^p(D_\sigma^*)$ as spaces of signals. We propose a common point of viev on $E^p(D_\sigma)$ and $E^p(D_\sigma^*)$.

Theorem 2. *Function f belongs to $E^2(D_\sigma^*)$ if and only if, when the function*

$$F(w) = f\left(-w + \frac{2\sigma i}{\pi} \cos \frac{w\pi i}{2\sigma}\right) \sqrt{-1 + \sin \frac{w\pi i}{2\sigma}},$$

where $\sqrt{1} = 1$, belongs to $E^2(D_\sigma)$.

The proof of the theorem is based on the following lemma.

Lemma 3. *Function*

$$\tilde{w} = -w + \frac{2\sigma i}{\pi} \cos \frac{w\pi i}{2\sigma}$$

comformally maps D_σ into D_σ^*.

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On the equivalency classes of weakly conjugated inner mappings

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A map is called inner mapping if it is open and isolated (the preimage of a point consists of isolated points). Yuriy Trokhimchuk studied inner mappings a lot during his life and published a book [5]. Topological properties of dynamical systems generated by inner mappings were studied in [3].

the open question of topological dynamics of inner mappings of surfaces is whether there exists a class of structurally stable inner mappings. Conjugacy with a homeomorphism as a topological equivalence of inner mappings seems too strict to produce a structurally stable map. It is proven in [1, 2] for Anosov endomorphisms. A paper [4] produced some examples even for the wandering set. It seems that indeed there is no structurally stable inner mapping up to topological conjugacy.

In that case it seems reasonable to find another definition of the topological equivalence such that it allows structural stability. Possible candidates are discussed.

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Extremal problem for non-overlapping domains with free poles

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Let $\mathbb{N}$ and $\mathbb{R}$ be the sets of natural and real numbers, respectively, $\mathbb{C}$ be the complex plane, $\overline{\mathbb{C}} = \mathbb{C}\bigcup\{\infty\}$ be the Riemann sphere, and $r(B, a)$ be the inner radius of the domain $B \in \overline{\mathbb{C}}$ with respect to the point $a \in B$.

Consider the following problem which was formulated in 1994 [1].

Problem 1. Consider the product

$$I_n(\gamma) = r^{\gamma}\left(B_0, 0\right) \prod_{k=1}^{n} r\left(B_k, a_k\right),$$

where B_0, B_1,...,B_n $(n \geq 2)$ are pairwise non-overlapping domains in $\overline{\mathbb{C}}$ and $a_0 = 0$ and $|a_k| = 1$ for $k = \overline{1, n}$, and $0 < \gamma \leqslant n$. Show that it attains its maximum at a configuration of domains B_k and points a_k possessing rotational n-symmetry.

This problem has a solution only if $\gamma \leqslant n$ as soon as $\gamma = n+\epsilon$, $\epsilon > 0$, the problem has no solution. Currently it still unsolved in general, only partial results are known [2].

The following theorem holds [3].

Theorem 2. *Let $n \in \mathbb{N}$ and $n \geqslant 2$. Then for any $\beta \in (0; \frac{1}{2}]$ there exists $n_0(\beta)$ such that for all $n \geqslant n_0(\beta)$ and for all $\gamma \in (1, n^{\beta}]$ and for any different points of a unit circle and for any different system of non-overlapping domains B_k, such that $a_k \in B_k \subset \overline{\mathbb{C}}$ for $k = \overline{1, n}$, and $a_0 = 0 \in B_0 \subset \overline{\mathbb{C}}$, the following inequality holds*

$$r^{\gamma}\left(B_0, 0\right) \prod_{k=1}^{n} r\left(B_k, a_k\right) \leqslant \left(\frac{4}{n}\right)^n \frac{\left(\frac{4\gamma}{n^2}\right)^{\frac{\gamma}{n}}}{\left(1 - \frac{\gamma}{n^2}\right)^{n+\frac{\gamma}{n}}} \left(\frac{1 - \frac{\sqrt{\gamma}}{n}}{1 + \frac{\sqrt{\gamma}}{n}}\right)^{2\sqrt{\gamma}}. \tag{1}$$

Equality is attained if a_k and B_k for $k = \overline{0, n}$, are, respectively, poles and circular domains of the quadratic differential

$$Q(w)dw^2 = -\frac{(n^2 - \gamma)w^n + \gamma}{w^2(w^n - 1)^2}\, dw^2.$$

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Inner harmonic measure for the fractional Laplacian

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The talk is based on [9], and it deals with the theory of potentials with respect to the α-Riesz kernel $|x-y|^{\alpha-n}$ of order $\alpha \in (0,2]$ on $\mathbb{R}^n$, $n \geqslant 3$. We first focus on the inner α-harmonic measure ε_y^A for $A \subset \mathbb{R}^n$ arbitrary, being motivated by the known fact that it is the main tool in solving the generalized Dirichlet problem for α-harmonic functions (see [1, 7]). Here ε_y is the unit Dirac measure at $y \in \mathbb{R}^n$, and μ^A the inner α-Riesz balayage of a Radon measure μ to $A \subset \mathbb{R}^n$ (see [8], cf. also [4] where $\alpha = 2$).

We describe the Euclidean support of the inner α-harmonic measure ε_y^A, provide a formula for evaluation of its total mass $\varepsilon_y^A(\mathbb{R}^n)$, establish the vague continuity of the map $y \mapsto \varepsilon_y^A$ outside the inner α-irregular points for A, and obtain necessary and sufficient conditions for ε_y^A to be of finite energy (more generally, for ε_y^A to be absolutely continuous with respect to inner capacity) as well as for $\varepsilon_y^A(\mathbb{R}^n) \equiv 1$ to hold. Those criteria are given in terms of newly defined concepts of inner α-thinness and inner α-ultrathinness of A at infinity (see [9]) that for $\alpha = 2$ and A Borel coincide with the concepts of outer 2-thinness at infinity by Doob [5] and Brelot [2], respectively.

Further, we extend some of these results to μ^A general by verifying the integral representation formula for inner balayage:

$$\mu^A = \int \varepsilon_y^A \, d\mu(y).$$

We also show that for every $A \subset \mathbb{R}^n$, there exists a K_σ-set $A_0 \subset A$ such that

$$\mu^A = \mu^{A_0} \quad \text{for all } \mu,$$

and give various applications of this theorem. In particular, we prove the vague and strong continuity of the inner swept, resp. inner equilibrium, measure under an approximation of A arbitrary, thereby strengthening Fuglede's result [6], established for A Borel.

Being mainly new even for $\alpha = 2$, the results obtained also present a further development of the theory of inner Newtonian capacities and of inner Newtonian balayage, originated by Cartan [3, 4].

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Мультиплікатори в просторах Харді та пов'язаних з ними просторах

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Нехай m — деяке натуральне число, $\mathbb{C}^m$ — множина впорядкованих наборів комплексних чисел $\mathbf{z} = (z_1, \ldots, z_m)$. Через $D^m = \{\mathbf{z} \in \mathbb{C}^m : |z_j| < 1,\, 1 \le j \le m\}$ позначимо одиничний полікруг з кістяком $T^m = \{\mathbf{z} \in \mathbb{C}^m : |z_j| = 1,\, 1 \le j \le m\}$. Через $H_1(D^m)$ позначимо множину аналітичних в полікрузі D^m функцій f, для яких виконується умова

$$\|f\|_{H_1(D^m)} = \sup_{0<r_j<1, 1\le j\le m} \int_0^{2\pi} dt_1 \ldots \int_0^{2\pi} |f(r_1 e^{it_1}, \ldots, r_m e^{it_m})| dt_m < \infty.$$

Відмітимо, що при $m = 1$ отримаємо звичайні одновимірні класи Харді $H_1(D)$ і в цьому випадку верхній індекс будемо опускати.

За допомогою послідовності комплексних чисел $\Lambda = \{\lambda_k\}, k \in \mathbb{Z}_+$ кожній $f \in H_1(D^m)$ з рядом Тейлора $f(z) = \sum_{\nu=1}^{\infty} F_\nu(z)$, $F_\nu(z) = \sum\limits_{k_1+\ldots+k_m=\nu} \mathbf{c_k} z^{\mathbf{k}}$ поставимо у відповідність функцію $\Lambda f(z) = \sum_{\nu=1}^{\infty} \lambda_\nu F_\nu(z)$ та означитимо наступним чином мультиплікатор. Послідовність комплексних чисел Λ називається мультиплікатором, що діє з $H_1(D^m)$ в $H_1(D^m)$, якщо $\|\Lambda f\|_{H_1(D^m)} \le M\|f\|_{H_1(D^m)}$.

З класичними класами Харді тісно пов'язані дійсні класи Харді. Під дійсним класом Харді ReH_1 розуміють простір функцій $F : R \to R$, що є дійсними частинами граничних значень функцій $f \in H_1(D)$ $F(t) = \lim\limits_{r\to 1} Ref(re^{it})$.

Дійсний клас Харді є банаховим простором з нормою $\|F\|_{ReH_1} = \|F\|_{L_1} + \|\overline{F}\|_{L_1}$, де $\overline{F}$ — функція спряжена до F, L_1 — простір сумовних функцій з нормою $\|F\|_{L_1} = \int\limits_0^{2\pi} |F(x)|dx$.

Аналогічно, послідовність $\Lambda = \{\lambda_k\}, k \in \mathbb{Z}_+$ називається мультиплікатором з ReH_1 в ReH_1, якщо для $F \in ReH_1$ з рядом Фур'є $F(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kt + b_k \sin kt$ ряд $\Lambda F(x) \sim \frac{\lambda_0 a_0}{2} + \sum_{k=1}^{\infty} \lambda_k(a_k \cos kt + b_k \sin kt)$ є рядом Фур'є деякої функції $\Lambda F \in ReH_1$, тобто $\|\Lambda F\|_{ReH_1} \le M\|F\|_{ReH_1}$.

Теорема 1. *Для того щоб послідовність комплексних чисел $\Lambda = \{\lambda_k\}$ була мультиплікатором з простору $H_1(D)$ в $H_1(D)$, необхідно і достатньо, щоб існувала така послідовність $\mu_k \in \mathbb{C}$ така, що* $\sup\limits_n \int_0^{2\pi} \left| \sum_{k=0}^n \lambda_k e^{-ikt} + \sum_{k=1}^n \mu_k e^{ikt} \right| dt < \infty$

Теорема 2. *Для того щоб послідовність комплексних чисел $\Lambda = \{\lambda_k\}$ була мультиплікатором з простору $H_1(D^m)$ в $H_1(D^m)$, необхідно і достатньо, щоб існувала така послідовність $\mu_k \in \mathbb{C}$, що* $\sup\limits_n \int_0^{2\pi} \left| \sum_{k=0}^n \lambda_k e^{-ikt} + \sum_{k=1}^n \mu_k e^{ikt} \right| dt < \infty.$

Теорема 3. *Для того щоб послідовність дійсних чисел* $\Lambda = \{\lambda_k\}$ *була мультиплікатором з простору* ReH_1 *в* ReH_1, *необхідно і достатньо, щоб існував такий розклад* $\lambda_k = \alpha_k + \beta_k$, $\alpha_k, \beta_k \in \mathbb{R}$, *що* $\sup\limits_n \int_0^{2\pi} \Big| \sum_{k=0}^n \alpha_k \cos kx + \beta_k \sin kx \Big| dx < \infty$

Про деякі закономірності квазі-геодезичних відображень узагальнено-рекурентних просторів

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Нехай узагальнено-рекурентний простір параболічного типу [3] (V_n, g_{ij}, F_i^h) допускає нетривіальне квазі-геодезичне відображення [1] на псевдоріманововий простір $(\overline{V}_n, \overline{g}_{ij})$. Тоді в сумісній за відображенням системі координат (x^i) виконуються основні рівняння [3]

$$\overline{\Gamma}_{ij}^h(x) = \Gamma_{ij}^h(x) + \psi_{(i}(x)\delta_{j)}^h + \phi_{(i}(x)F_{j)}^h(x),$$

$$F_{ij} = -F_{ji}, \quad F_{ij} = g_{i\alpha}F_j^\alpha, \quad \overline{F}_{ij} = -\overline{F}_{ji}, \quad \overline{F}_{ij} = \overline{g}_{i\alpha}F_j^\alpha,$$

$$F_\alpha^h F_i^\alpha = 0$$

$$F_{(i,j)}^h = F_{(i}^h q_{j)}.$$

Тут "," - знак коваріантної похідної відносно зв'язності Γ в V_n.

Розглянуто випадок, коли узагальнено-рекурентний простір параболічного типу з інтегровною афінорною структурою (V_n, g_{ij}, F_i^h) допускає квазі-геодезичне відображення зі збереженням вектора узагальненої рекурентності [3], отже в просторі $(\overline{V}_n, \overline{g}_{ij})$ для афінора F_i^h виконуються співвідношення

$$F_{(i|j)}^h = F_{(i}^h q_{j)},$$

де "|" - знак коваріантної похідної відносно зв'язності $\overline{\Gamma}$ в V_n.

Зауважимо, що образ узагальнено-рекурентного простору при квазі-геодезичному відображенні необхідно буде також узагальнено-рекурентним простором [3], але збереження вектора узагальненої рекурентності при цьому не є необхідним.

За таких умов отримано нову форму основних рівнянь [2] квазі-геодезичних відображень узагальнено-рекурентних просторів параболічного типу, яка допускає ефективне дослідження.

Побудовано перетворення, яке дає змогу із пари узагальнено-рекурентних просторів, що знаходяться в квазі-геодезичному відображенні зі збереженням вектора узагальненої рекурентності, отримати нову пару узагальнено-рекурентних просторів, що також знаходяться в квазі-геодезичному відображенні.

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Геодезические отображения римановых пространств второго приближения

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Рассмотрим риманово пространство V_n, отнесенное к произвольной системе координат x^h. В окрестности произвольной фиксированной точки $M_0(x_0^h)$ строим пространство $\tilde{V}_n^2$, которое реализует приближение второго порядка для V_n, с метрическим тензором $\tilde{g}_{ij}(y)$ [1]:

$$\tilde{g}_{ij}(y) = \underset{o}{g_{ij}} + \frac{1}{3}\underset{o}{R}_{i\alpha\beta j}y^\alpha y^\beta, \tag{1}$$

где

$$\underset{o}{g_{ij}} = g_{ij}(M_0), \underset{o}{R}_{i\alpha\beta j} = R_{i\alpha\beta j}(M_0).$$

Для исследования геодезических отображений пространства $\tilde{V}_n^2$ основные уравнения Н.С. Синюкова [2]

$$\begin{gathered}\tilde{\nabla}_k\tilde{a}_{ij} = \tilde{\lambda}_{(i}\tilde{g}_{j)k}\\ n\tilde{\nabla}_k\tilde{\lambda}_i = \tilde{\mu}\tilde{g}_{ik} + \tilde{a}_{\alpha i}\tilde{R}_k^\alpha - \tilde{a}_{\alpha\beta}\tilde{R}^{\alpha\beta}_{.ik.}\\ (n-1)\tilde{\nabla}_k\tilde{\mu} = 2(n+1)\tilde{\lambda}_\alpha\tilde{R}_k^\alpha + \tilde{a}_\alpha^\beta(2\tilde{\nabla}_\beta\tilde{R}_k^\alpha - \tilde{\nabla}_k\tilde{R}_\beta^\alpha)\end{gathered} \tag{2}$$

представлены в эквивалентном виде

$$\begin{gathered}\tilde{g}_{i\alpha}\frac{\partial\tilde{a}_j^\alpha}{\partial y^k} + \tilde{a}_j^\alpha\tilde{\Gamma}_{\alpha k,i} - \tilde{a}_i^\alpha\tilde{\Gamma}_{jk,\alpha} = \tilde{\lambda}_{(i}\tilde{g}_{j)k}\\ n\left(\frac{\partial\tilde{\lambda}^\alpha}{\partial y^i}\tilde{g}_{\alpha i} + \tilde{\lambda}^\alpha\tilde{\Gamma}_{\alpha j,i}\right) = \tilde{\mu}\tilde{g}_{ij} + \tilde{a}_i^\alpha\tilde{R}_{\alpha j} - \tilde{a}_\beta^\alpha\tilde{R}^\beta_{.ij\alpha}\\ (n-1)\tilde{\nabla}_k\tilde{\mu} = 2(n+1)\tilde{\lambda}^\alpha\tilde{R}_{\alpha k} + \tilde{a}_\alpha^\beta(2\tilde{\nabla}_\beta\tilde{R}_k^\alpha - \tilde{\nabla}_k\tilde{R}_\beta^\alpha)\end{gathered} \tag{3}$$

Исследуя уравнения (3), компонент тензора $\tilde{a}_{ij}(y) = \tilde{g}_{i\alpha}\tilde{a}_j^\alpha$, вектора $\tilde{\lambda}_i = \tilde{\lambda}^\alpha\tilde{g}_{\alpha i}$ и функции $\tilde{\mu}(y)$ получены в виде степенных рядов, коэффициенты которых определены значениями объектов пространства V_n в точке M_0. Изучается вопрос сходимости полученных рядов.

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